1. (8 points) Assume there are three major computer vision algorithms that are designed to detect objects. 45% of researchers prefer using algorithm A, 40% researchers prefer using algorithm B, 15% prefer using algorithm C. For a certain test, algorithm A has a precision of 70%, algorithm B has a precision of 90%, and algorithm C has a precision of 55%. Precision here refers to the probability that an algorithm leads to correct results.

(a) (5 points) What is the probability that a random researcher has a correct result in this test using his preferred algorithm?

Solution: Let A, B and C denote the event that a researcher prefers using algorithm A, B and C respectively. Let P denote the event that an algorithm is precise in this test (yields correct result).

$P(P) = P(A) \cdot P(P|A) + P(B) \cdot P(P|B) + P(C) \cdot P(P|C) = 0.45 \times 0.7 + 0.4 \times 0.9 + 0.15 \times 0.55 = 0.7575$

(b) (3 points) A researcher conducts the test using his preferred algorithm, and yields correct result. What is the probability that this researcher is using algorithm A?

Solution:

$$P(A|P) = \frac{P(A) \cdot P(P|A)}{P(P)} = \frac{0.45 \times 0.7}{0.7575} = 0.4158$$

2. (8 points)

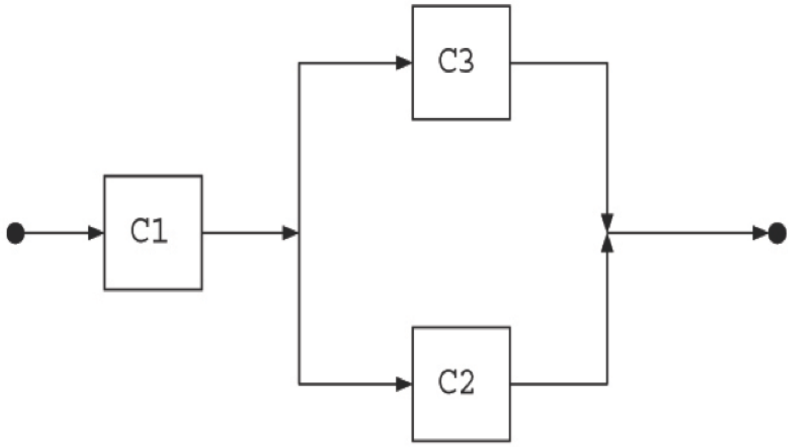

FIGURE 1. System configuration

(a) (4 points) What is the reliability for the system configuration shown in Fig.1? Assume the reliabilities for components C1, C2, and C3 are respectively p_1, p_2, and p_3. (Note that the interpretation of reliability here is the probability that, at a random time, the component is operational. The reliabilities of components are all mutually independent. There is no "lifetime" of the system in this view)

(b) (4 points) Now assume that we have lifetime distributions $F_1(t), F_2(t), F_3(t)$ respectively for C1, C2 and C3, i.e. $F_1(t) = P(T_1 < t)$ denotes the probability that the lifetime of C1 (T_1) is shorter than t. Give an expression in terms of $F_1(t), F_2(t), F_3(t)$ for the distribution of the time until the system fails $F(t)$ (i.e. the lifetime of the whole system).

Solution:

(a) $p_1\big[1-(1-p_2)(1-p_3)\big]$

(b) $F(t) = F_1(t) + F_{23}(t) - F_1(t)F_{23}(t) = F_1(t) + F_2(t)F_3(t) - F_1(t)F_2(t)F_3(t)$

3. (8 points) A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$ respectively, and is received incorrectly with probability ϵ_0 and ϵ_1 respectively (see figure below). Errors in different symbol transmissions are independent.

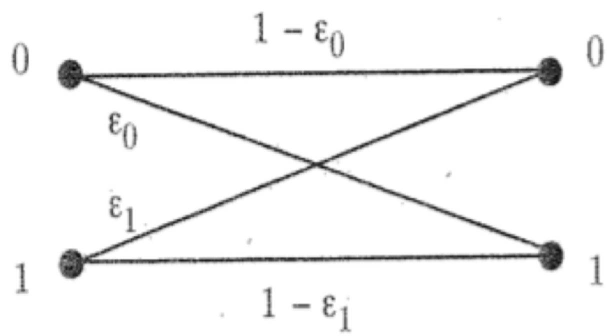

FIGURE 2. Error probabilities in a binary communication channel

(a) (3 points) What is the probability that a certain symbol is received correctly?
(b) (2 points) A string of symbols "1011" is sent. What is the probability that it is received correctly?
(c) (2 points) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. For example, a 0 is transmitted as 000 (or a 1 is transmitted as 111), and is decoded as 0 if and only if the received three-symbol string contains at least two 0s. What is the probability that 0 is received correctly?
(d) (1 points) For what values of ϵ_0 is there an improvement in the probability of correct decking of a 0 when the scheme in part (c) is used?

Solution:

(a) Let X denote the symbol sent, and Y denote the symbol received.

$$\begin{aligned} P(X=Y) &= P(X=0, Y=0) + P(X=1, Y=1) \\ &= P(X=0) \cdot P(Y=0|X=0) + P(X=1) \cdot P(Y=1|X=1) \\ &= p \cdot (1-\epsilon_0) + (1-p) \cdot (1-\epsilon_1) \end{aligned}$$

(b) $(1-\epsilon_1)(1-\epsilon_0)(1-\epsilon_1)(1-\epsilon_1)$

(c) 0 is sent as 000, and is received correctly if one of the following strings are received: 000, 100, 010, 001. So the probability is:

$$(1-\epsilon_0)^3 + 3(1-\epsilon_0)^2\epsilon_0$$

(d) There's an improvement when

$$(1-\epsilon_0)^3 + 3(1-\epsilon_0)^2\epsilon_0 > 1-\epsilon_0$$

i.e. when $\epsilon_0 < 0.5$.

4. (8 points) A computer system consists of n subsystems, each of which has an exponential distribution of lifetime with parameter $\lambda_i, i = 1, 2, ..., n$. Each subsystem is independent but the whole computer system fails if any of the subsystems do. Find the distribution of the minimum of these lifetimes.

Solution: [From homework 4] Let X_i be a random variable that denotes the lifetime of the i-th component. We consider $Y = \min(X_1, X_2, ..., X_n)$. So,

$$P(Y < y) = 1 - P(X_1 > y \cap X_2 > y \cap ... \cap X_n > y) = 1 - \prod_{i=1}^{n} P(X_i > y)$$

As we know that $X_1, ..., X_n$ follows exponential distribution, i.e.

$$P(X_i < x) = F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \leq 0 \\ 0, & x < 0 \end{cases}$$

So,

$$F(Y) = P(Y < y) = 1 - \prod_{i=1}^{n} e^{-\lambda_i y} = 1 - e^{-(\sum_{i=1}^{n} \lambda_i)y}$$

So, Y follows an exponential distribution with parameter $\sum_{i=1}^{n} \lambda_i$.

5. (8 points) A *series* system is composed of two components with lifetime distribution $\mathrm{Exp}(\lambda_1)$ and $\mathrm{Exp}(\lambda_2)$ respectively. What is the probability that component 2 causes the system failure, assuming that the component lifetimes are independent.

Solution: Refer to course reader Example 4.9 on page 87.

6. (5 points) Let $\{X_i, i = 1, ..., r\}$ be r independent identically distributed (iid) random variables with Exp(λ) distribution, and let $Y = \sum_{i=1}^{r} X_i$.

(a) (3 points) Derive the Laplace transform of the PDF of X_i.

(b) (3 points) Derive the Laplace transform of the PDF of Y.

Solution: Refer to course reader Example 4.7 on page 85.