Physics 1BH Winter 2022 Final Exam

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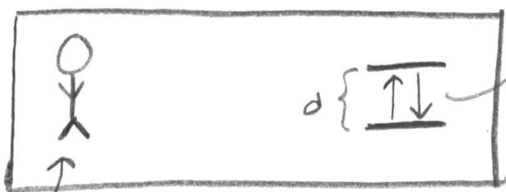
Instructions

- This exam is 3 hours long.
- This exam is closed-notes, closed-book. No phones or calculators allowed.
- The standard list of equations is included.
- You will independently be given points for your reasoning, your mathematical work, and for correctness of answers, so make sure to show your reasoning and work even if you're not certain you can compute the correct answer. Try to convey your solution strategy even if you can't execute it in time.
- If on a given problem you need some space to do scratch work, do that scratch work **ELSEWHERE**. Only the work that is included on this test packet **in the boxed space provided will be graded**. If you need additional sheets please ask the instructor/proctor.
- **Put boxes around any final numerical or symbolic answers.**
- **DO NOT BOX MULTIPLE "FINAL" ANSWERS. If you do not want an answer graded, cross it out COMPLETELY with an 'X.'** Providing multiple "final" answers will result in a zero for that question.

Problem 1	15 points	Problem 6	20 points
Problem 2	30 points	Problem 7	30 points
Problem 3	20 points	Problem 8	15 points
Problem 4	30 points	Problem 9	20 points
Problem 5	20 points		

1. (15 points) At a graduation party, your friend's grandmother finds out that you study physics and asks you to explain the concept of time dilation to her. She is a retired electrical engineer and so can easily follow any mathematics or geometric diagrams. Using the diagram(s) of the moving "light clock" discussed in class, derive the time dilation relation for intervals of time as measured by two observers, one stationary and the other moving at speed $v \neq c$.



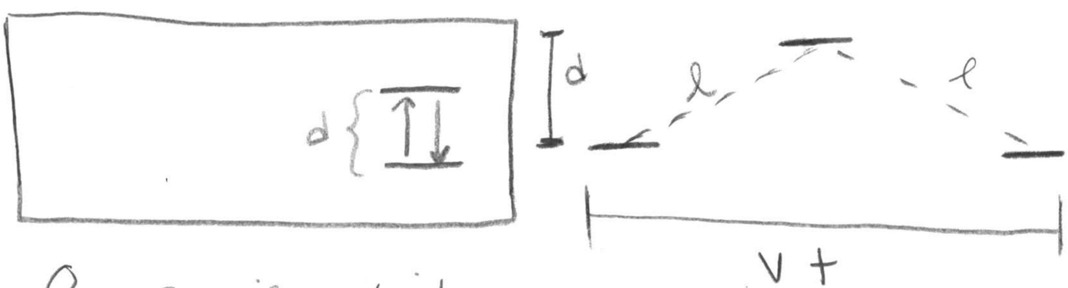
light is flashed up and returns back down

$v = d/t \rightarrow t = d/v$

$t' = \frac{2d}{c}$

$\hookrightarrow d = \frac{ct'}{2}$

Ann is inside the train car with the "light clock"



Bob is outside the train car watching the light move

(where v is the speed the train car is moving at

$l = \sqrt{d^2 + v^2t^2/4}$

$\hookrightarrow t = \frac{2l}{c} = \frac{2\sqrt{d^2 + \frac{v^2t^2}{4}}}{c}$

$t = \frac{2\sqrt{\frac{c^2t'^2}{4} + \frac{v^2t^2}{4}}}{c}$

$t^2 = \frac{\cancel{4}\left(\frac{c^2t'^2 + v^2t^2}{\cancel{4}}\right)}{c^2}$

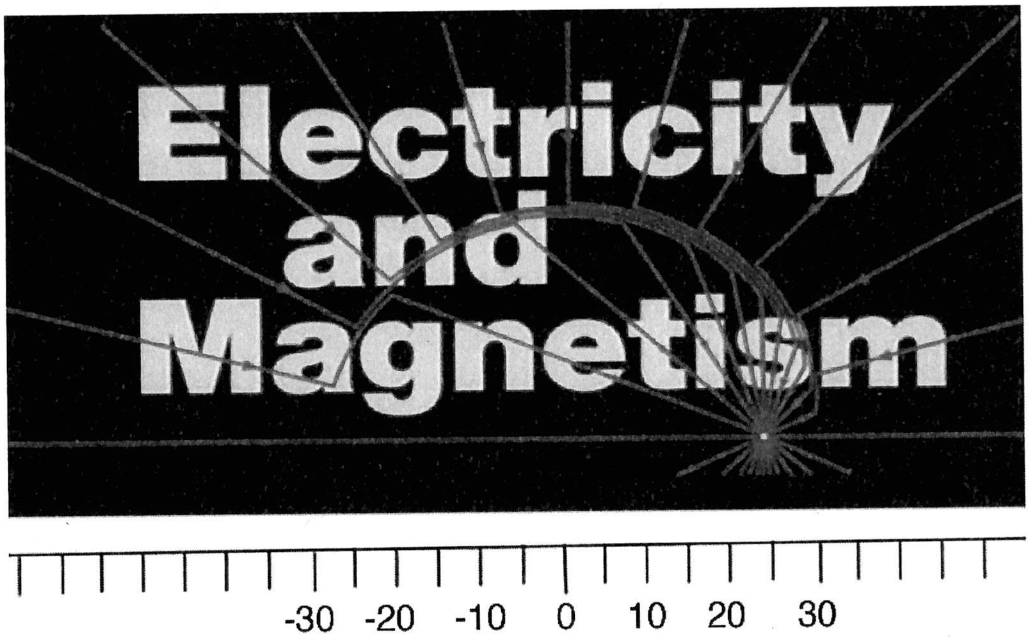
$c^2t^2 = c^2t'^2 + v^2t^2$

$\frac{c^2t^2 - v^2t^2}{c^2} = t'^2$

$t' = \sqrt{t^2(1 - v^2/c^2)}$

$\hookrightarrow t' = t\sqrt{1 - v^2/c^2}$

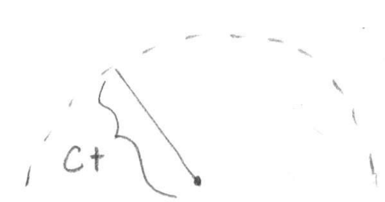
2. (30 points) The figure below is from the cover of your book. Distances in centimeters have been added with tick marks. To keep numbers simple, the **speed of light is $c = 30$ cm/ns** (ns = nanosecond).
You (at rest) see an **electron** at time $t = 1.0$ **ns** moving along the x-axis and the associated electric field lines at that instant. The outermost (radial) field lines represent a spherically symmetric electric field and continue straight out to infinite distance. Except for sudden acceleration(s), the electron only moves at a constant speed. (Assume the tick marks are equally spaced and align with what you need to read off.)



a) (5 points) What was the electron's speed just before $t = 0$?

The elecctron's speed just before t=0 was 0 m/s because the outer most field lines are those of a still particle.

2b) (10 points) Using the diagram, find the electron's constant velocity (in units of c) at the moment shown at $t = 1.0$ ns and from this calculate the Lorentz factor, γ.



$ct = 30$ cm
$vt = 25$ cm
$t' = t\sqrt{1 - v^2/c^2}$

$\frac{30\text{ cm}}{c} = \frac{25\text{ cm}}{v}$

$v = \frac{25\text{cm}}{30\text{cm}} c$

$\boxed{v = \frac{5}{6} c}$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 25/36}}$

$\boxed{\gamma = \sqrt{\frac{36}{11}}}$

$v^2 = \frac{25}{36} c^2$

$\frac{25}{36} \cancel{c^2} / \cancel{c^2}$

2c) (15 points) At the moment shown in the figure, what is the magnitude of the electric field **at the origin**? Leave your answer in units of $e/4\pi\epsilon_0$ (i.e. $E = (e/4\pi\epsilon_0)\alpha$, where your goal is to find the number α).

Hint: To a moving observer, the electron is at rest, where you know how to easily calculate the field. But think carefully what distances (if any) are contracted!

$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$, $Q = -e$, $r = ?$

$\hat{r} = -$

$\rightarrow$ if electron appears to be at rest

length contraction:

$L = L'/\gamma$ (L' is inside reference frame)

$L' = \gamma L$, $L = 25$ cm $= .25$ m

$\rightarrow L' = \sqrt{\frac{36}{11}}\ .25$ m

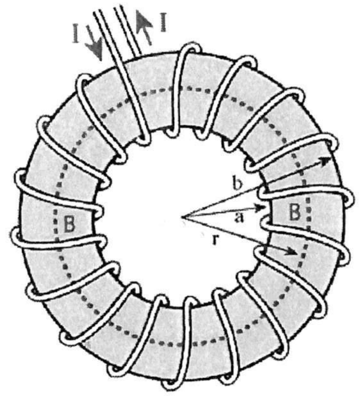
$E = \frac{1}{4\pi\epsilon_0} \frac{e}{L'^2}$

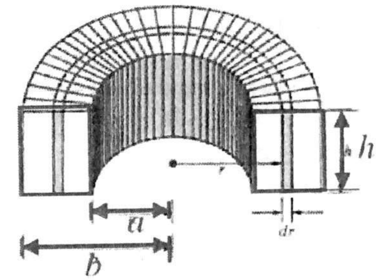
$= \frac{1}{4\pi\epsilon_0} \frac{e}{.25^2 (36/11)}$

$= \frac{1}{4\pi\epsilon_0} \frac{\cancel{16}\,4\,e}{1} \cdot \left(\frac{11}{\cancel{36}\,9}\right)$

$\boxed{E = \frac{e}{4\pi\epsilon_0} \cdot \frac{44}{9}}$

3. (20 points) The toroid pictured has N turns and a rectangular loop area, $A = h(b - a)$. A current I flows through the wire at a **steady-rate** and sets up a **static** magnetic field $\boldsymbol{B}$.





a. (10 points) Find the magnetic field strength, $\boldsymbol{B}(r)$, as a function of radial distance, r, from the radial axis of the toroid. In what direction (clockwise/counterclockwise) does the magnetic field point?

The magnetic field is going clockwise inside the toroid

Ampere's Law:

$B \cdot 2\pi r = I_{enc}\,\mu_0$, $I_{enc} = IN$

$I = \frac{Q}{t}$

$$\boxed{B(r) = \frac{IN\mu_0}{2\pi r}}$$

3b. (10 points) What is the total self-inductance, L, of the toroid?
Hint: Unlike the solenoid, here the magnetic field depends on r.

$$L = \frac{\Phi_B}{I}, \quad \Phi_B = \vec{B} \cdot A$$

$$= \frac{I N \mu_0}{2\pi r} \cdot h(b-a)$$

$$L = \frac{\cancel{I} N \mu_0 \cdot h(b-a)}{2\pi r} \cdot \frac{1}{\cancel{I}}$$

$$\boxed{L = \frac{N \mu_0}{2\pi r} h(b-a)}$$

4. (30 points) A spherical insulator extends from the origin out to a radius a and has a charge distribution given by $\rho(r) = \rho_0 \left(\frac{r}{a}\right)^3$. Beyond radius a (outside of the sphere), is the vacuum of empty space.

(15 points)

a. What is the electric field **inside** of the sphere, at radial distances $r < a$?

b. What is the electric field **outside** of the sphere, at radial distances $r > a$?

Gauss's Law:

$$\oint E \cdot dA = \frac{Q_{enc}}{\varepsilon_0}$$

$$Q_{enc} = \int_0^r \rho_0 \left(\frac{r'}{a}\right)^3 dr' = \frac{\rho_0}{a^3}\int_0^r r'^3\, dr' = \frac{\rho_0}{a^3}\left[\frac{r'^4}{4}\right]_0^r = \frac{\rho_0 r^4}{4a^3}$$

$$E \cdot 4\pi \cancel{r^2} = \frac{\rho_0 r^{\cancel{4}2}}{4a^3}$$

inside sphere

$$\boxed{E = \frac{\rho_0 r^2}{16\pi a^3} \text{ pointing radially outward}}$$

Q_{enc} when $r > a$ $= \frac{\rho_0 a}{4}$ (only charge inside of sphere)

$$E \cdot 4\pi r^2 = \frac{\rho_0 a}{4}$$

$$\boxed{E = \frac{\rho_0 a}{16\pi r^2} \text{ pointing radially outward}}$$

outside sphere

4c. (15 points) How much work was required to assemble the charge distribution $\rho(r) = \rho_0 \left(\frac{r}{a}\right)^3$?

Hint: There are multiple ways to calculate this. One method involves using your answers in 2a and 2b.

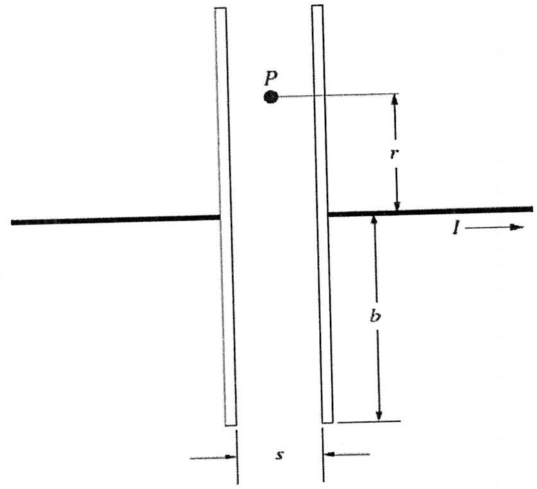
$$W = \frac{\epsilon_0}{2} \int E^2 dV$$

$$= \frac{\epsilon_0}{2} \int_0^r \frac{\rho_0^2 r'^4}{16\pi a^3} dr'$$

$$= \frac{\epsilon_0 \rho_0^2}{32\pi a^3} \left[\frac{r'^5}{5} \right]_0^r$$

$$\boxed{W = \frac{\epsilon_0 \rho_0^2 r^5}{160\pi a^3} \text{ J}}$$

5. (20 points) The capacitor shown below has circular plates with radii b. It is being **discharged** with a constant current I. The plates are separated by a very small distance, s, which is much smaller than b. Point P is a distance r from the axis of the attached wires and $r < b$, where edge effects can be safely ignored.



a) (10 points) Prove that the total "displacement current," I_D, between the plates is equal to the conduction current in the wires, I.

conduction current:

$$\vec{B} \cdot 2\pi r = I_{enc}\mu_0 + \underbrace{\mu_0 \epsilon_0 \frac{\partial E}{\partial t}}_{0}$$

$$I = \frac{B \cdot 2\pi r}{\mu_0}$$

displacement current:

E-field in a capacitor $\rightarrow E = \sigma/\epsilon_0$

$$Q = C\Delta\phi \rightarrow \sigma = \frac{C\Delta\phi}{\pi r^2}$$

$$E = -\nabla\phi \rightarrow -\nabla\phi = \frac{C\Delta\phi}{\epsilon_0 \pi r^2}$$

$$\vec{B} \cdot 2\pi r = \underbrace{I_{enc}\mu_0}_{0} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

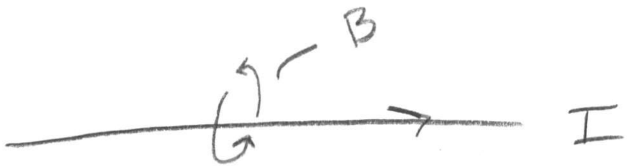
$$\mu_0 \epsilon_0 \frac{\partial E}{\partial t} = I\mu_0$$

$$I = \epsilon_0 \frac{\partial E}{\partial t} = \frac{B \cdot 2\pi r}{\mu_0}$$

5b) (10 points) What is the magnetic field at point P as a function of r? In what direction does it point (up/down/left/right/out- or into-the-page)?

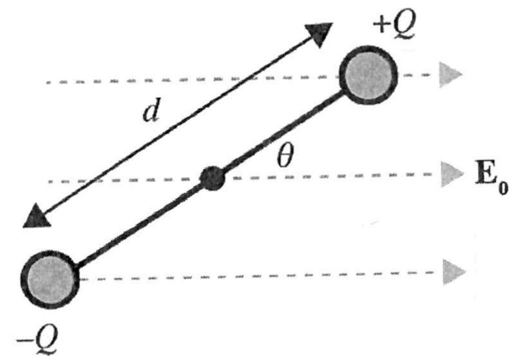
$$|B(r)| = \frac{I\mu_0}{2\pi r}$$

it points out of the page



so at P it points out of the page

6. (20 points) Two charged bodies of opposite charge $\pm Q$ and equal mass m are attached by a massless rod of length d that is free to rotate about its center, which is fixed in space. (The rotation is in the x-y plane of the page about the center.) A constant electric field, $\mathbf{E} = E_0\hat{x}$, is oriented in the x-direction with the charged bodies making an angle of θ with the field.



Recall that torque is given by $\tau = I\ddot{\theta} = \sum_i r_i F_i \sin\theta_i$, where I is the moment of inertia ($I = \sum_i m r_i^2$), m is mass, r_i is the distance from each mass to the axis of rotation, F_i is the force acting on each mass mass, and θ_i is the angle between F_i and r_i.

For small oscillations ($\sin\theta \approx \theta$), find the frequency of oscillation in terms of Q, m, E_0, d. In what direction (clockwise/counterclockwise) does the rod rotate?
Hint: What is the force on each charged mass?

The rod will start to rotate clockwise, then switch off between counterclockwise and clockwise.

$$\tau = \frac{d}{2} E_0 Q \sin\theta + \frac{d}{2} E_0 Q \sin\theta$$

$$\tau = d E_0 Q \sin\theta, \quad \sin\theta \approx \theta$$

$$2m\left(\frac{d}{2}\right)^2 \ddot{\theta} = d E_0 Q \theta$$

$$\frac{md}{2}\ddot{\theta} = E_0 Q \theta$$

$$\ddot{\theta} - \frac{2E_0 Q}{md}\theta = 0 \; \Big\} \; \omega = \frac{2E_0 Q}{md} \rightarrow \boxed{f = \frac{E_0 Q}{\pi m d}}$$

7. (30 points) While camping in a remote location, you realize that you forgot your phone charger. You want to build an AC generator to power your phone in case of an emergency. You decide to make a generator using the Earth's magnetic field. You need to provide a sinusoidal voltage source with an **RMS voltage 120 V**. You make a square loop with 2000 turns of wire. The Earth's magnetic field ***parallel*** to the ground is 10^{-4} T pointing from South to North. You can turn the coil as fast as 10 revolutions per minute.

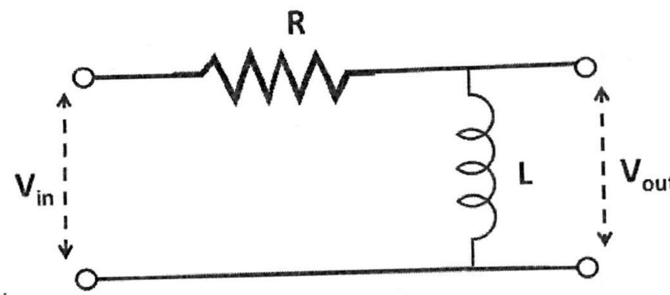
a) (5 points) You want the axis of rotation to be parallel to the ground for ease-of-use. What direction (North-South or East-West) is the axis of rotation best aligned with?

The axis of rotation should be aligned with the North - South direction so that the earth's magnetic field goes through the loop, and therefore when we turn it around the earth's magnetic field there will be a change in magnetic flux.

7b) (10 points) What size must the coil be? (It is square, so just specify one side's length.) To simplify your calculation use the following estimates: let $\pi \approx 3$, $\sqrt{2} \approx 1.5$.

7c) (15 points) Your generator provides 5W of power to the charger as you turn the loop. The induced current in the loop moves in the presence of Earth's magnetic field, creating a so-called "counter-torque" against your hand. Recalling the definition of torque in Problem 6, calculate the **maximum** value of this counter-torque.

8. (15 points) The circuit below is designed to mostly "pass" an AC voltage signal of certain frequencies placed at its input terminals (V_{in}), and block other types of frequencies. The resistor is 600Ω and the inductor is 500 μH.

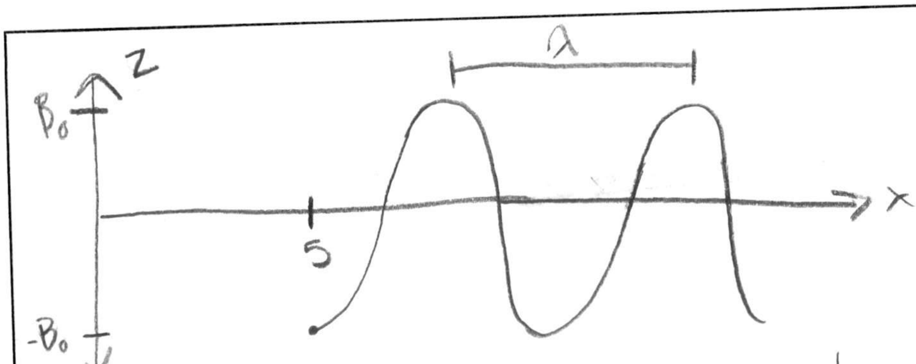


Calculate the ratio of the **squares** of the magnitudes of the output and input amplitudes, that is find $|V_{out}/V_{in}|^2$. By inspecting the behavior of this ratio as a function of frequency ω (low and high), describe whether this circuit is a low- or high-pass filter.
Hint: Any input/output devices connected to the terminals would complete the loops and allow for Kirchhoff's loop rule.

9. (20 points) An electromagnetic **plane wave** in free space has magnitude 3.0 V/m and is traveling along the **positive** $\hat{x}$ direction. This is a radio wave with frequency $f = 30$ MHz. The electric field points along the $\hat{y}$ direction. The magnetic field has its most negative value at $x = 5.0$ m at $t = 0$.

Write down the equation describing the magnetic field $\vec{B}(x,t)$ as a function of time and space coordinates in a form proportional to $\sin(kx - \omega t + \phi_0)$ or $\sin(kx + \omega t + \phi_0)$, where you specify the values of wavenumber, k, angular frequency, ω, and phase constant, ϕ_0.

Make sure you indicate the direction of the magnetic field. For this problem, signs matter! Put units on all numbers in your final answer, where radians are unitless.



$c = \lambda f$

$\lambda = \frac{c}{f}$

$\vec{B} = B_0 \sin(kx - \omega t + \phi_0)\hat{z}$,

↳ moving to right

$k = \frac{2\pi}{\lambda} = \frac{f(2\pi)}{c}$

$k = \frac{60\pi}{c} \frac{\text{MHz} \cdot \text{s}}{\text{m}}$

$\omega = \frac{f}{2\pi} = \frac{15}{\pi}$

$\vec{B}(x,t) = B_0 \sin\left(\frac{60\pi}{c} x - \frac{15}{\pi} + \phi_0\right)\hat{z}$

$\vec{B}(5,0) = \cancel{B_0}\sin\left(\frac{300\pi}{c} + \phi_0\right)\cancel{\hat{z}} = -\cancel{B_0}\cancel{\hat{z}}$

$\sin\left(\frac{300\pi}{c} + \phi_0\right) = -1$

$\frac{300\pi}{c} + \phi_0 = -\frac{\pi}{2}$

$\phi_0 = \frac{300}{c} - \frac{1}{2}$

$= \frac{600 - c}{2c}$

$\vec{B} = B_0 \sin\left(\frac{60\pi}{c} x - \frac{15}{\pi} + \frac{600-c}{2c}\right)\hat{z}$