Midterm 1
Physics 1AH, Fall 2021

Exam format:

- There are three problems on this exam, each with multiple parts. The point values for each part of each problem are indicated on the exam.
- You will have 1 hour to complete this exam.
- Please show your work on each problem.
- Unless stated otherwise, write all answers in terms of the symbols given in the problem and the physical constant $g \approx +9.8\ \mathrm{m/s^2}$.

Completing your exam:

- Write your name and UID on this page.
- Each part of each problem appears at the top of a page. This exam is printed double-sided, so check both the front and back of each page.
- Write all answers legibly inside the designated boxes. You will not need scratch paper.

Resources

- You may reference your optional double-sided 8.5-by-11 sheet of notes during the exam. You may not use any other resources (including print and electronic resources) during the exam. Calculators are not allowed.
- You may not communicate with anyone except the course staff during the exam.
- Some students will take this exam late. After completing the exam, do not discuss the exam with anyone who has not yet taken it.

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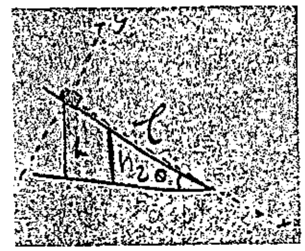
Problem 1 *(Three parts (a)-(c); 30 points)*

An object of mass m slides down a fixed and frictionless plane, which is inclined at an angle θ with respect to the horizontal. Assume that it starts from rest at height h from the ground.

Problem 1(a) *(10 points)*



Placing your coordinate system so that the x-axis is along the plane, and the origin is at the starting position of the object, find the equations of motion. Calculate the kinetic and potential energies of the system, as functions of time. Show that at any time, their sum is equal to a constant.

Space for working Problem 1(a)

$$\Sigma F_x = mg\sin\theta \qquad F_n = mg\cos\theta$$

$$\cancel{m}a = \cancel{m}g\sin\theta$$

$$\boxed{a = g\sin\theta}$$

$$\frac{dv}{dt} = g\sin\theta$$

$$\int \frac{dv}{g\sin\theta} = \int_0^t dt$$

$$\frac{1}{g\sin\theta}\cdot v(t) + V_0 = t\ , \quad V_0 = 0$$

$$\boxed{V(t) = gt\sin\theta}$$

Additional space for working Problem 1(a)

$$\int dx = \int_0^t gt\sin\theta \, dt$$

$$x(t) + x_0 = g\sin\theta \cdot \frac{t^2}{2}\ ; \quad x_0 = 0$$

$$\boxed{x(t) = \frac{1}{2} t^2 g\sin\theta}$$

At any time, the sum of kinetic and potential energy is a constant because no energy is leaving the system; there is no work done by friction.

$E_i = mgh$

$h = \ell\sin\theta$

$h/\ell = h_t/\ell - x(t)$

$$h_t = \frac{h(\ell - 1/2\, t^2 g\sin\theta)}{\ell}$$

$$\boxed{KE(t) = 1/2\, m g^2 t^2 \sin^2\theta}$$

$$U h_t = \frac{h\ell}{\ell} - \frac{\ell\sin\theta\, 1/2\, t^2 g\sin\theta}{\ell}$$

$$E = 1/2\, mg^2t^2\sin^2\theta + mgh - 1/2\, mg^2t^2\sin^2\theta$$

$$h_t = h - 1/2\, t^2 g\sin^2\theta$$

$$\boxed{U(t) = mg(h - 1/2\, g t^2 \sin^2\theta)}$$

$$\boxed{E(t) = mgh}$$

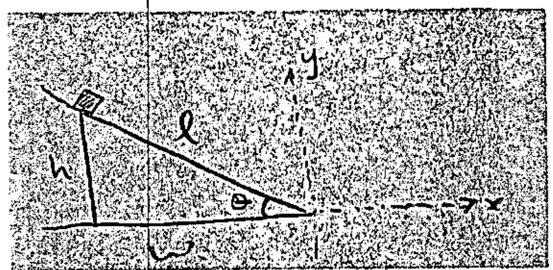
constant

Problem 1(b) starts on the next page→

An object of mass m slides down a fixed and frictionless plane, which is inclined at an angle θ with respect to the horizontal. Assume that it starts from rest at height h from the ground.

Problem 1(b) *(15 points)*

Place the coordinate system so that the y-axis is along the vertical direction, and the origin is at the bottom of the inclined plane. Find the equations of motion. Calculate the kinetic and potential energy of the system as functions of time, and show that their sum is a constant.



$\tan\theta = h/w$

$w = h\cot\theta$

Space for working Problem 1(b)

$F_n = mg\cos\theta$

$\Sigma F_y = F_n\cos\theta - mg$

$\cancel{m}a_y = mg\cos^2\theta - \cancel{m}g$

$a_y = g(\cos^2\theta - 1) = g\sin^2\theta$

$\int dV_y = \int_0^t g\sin^2\theta\, dt$

$V_y(t) = gt\sin^2\theta$

$\Sigma F_x = F_n\sin\theta$

$\cancel{m}a_x = \cancel{m}g\cos\theta\sin\theta$

$a_x = g\cos\theta\sin\theta$

$\int dV_x = \int_0^t g\cos\theta\sin\theta\, dt$

$V_x(t) = gt\cos\theta\sin\theta$

Additional space for working Problem 1(b)

$$\int dy = \int_0^t gt\sin^2\theta$$

$$y(t) - y(0) = \frac{1}{2}gt^2\sin^2\theta$$

$$\boxed{y(t) = \frac{1}{2}gt^2\sin^2\theta + h}$$

$$\int dx = \int_0^t gt\cos\theta\sin\theta\, dt$$

$$x(t) - x(0) = \frac{1}{2}gt^2\cos\theta\sin\theta$$

$$\boxed{x(t) = \frac{1}{2}gt^2\cos\theta\sin\theta - h\cot\theta}$$

$$KE(t) = \frac{1}{2}m\|v\|^2, \quad \|v\| = \sqrt{g^2t^2\sin^4\theta + g^2t^2\cos^2\theta\sin^2\theta}$$

$$= gt\sin\theta\sqrt{\sin^2\theta + \cos^2\theta}$$

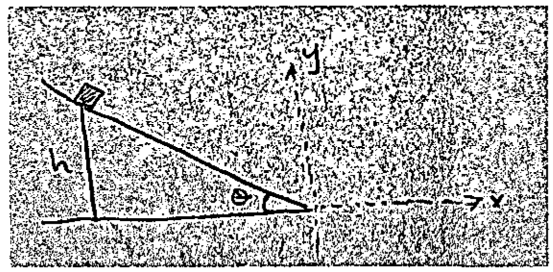
$$= gt\sin\theta$$

$$\boxed{KE(t) = \frac{1}{2}mg^2t^2\sin^2\theta}$$

$$h_t = h(1 - 1/2$$

Problem 1(c) starts on the next page→

An object of mass m slides down a fixed and frictionless plane, which is inclined at an angle θ with respect to the horizontal. Assume that it starts from rest at height h from the ground.



Problem 1(c) *(5 points)*
Using the same coordinate system as in (b), find the kinetic energy at the bottom (where the inclined plane meets the ground).

Space for working Problem 1(c)

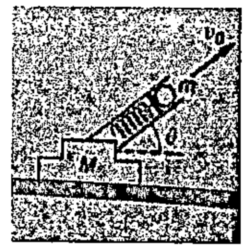
$\boxed{KE = mgh}$ $E_i = E_f$

Additional space for working Problem 1(c)

Problem 2 starts on the next page→

Problem 2 *(Two parts (a)-(b); 25 points)*

A spring gun fires a ball at an angle θ with respect to the horizontal. The mass of the ball is m, and the mass of the spring gun is M. The spring gun is initially at rest, and it is on a frictionless surface. At the time the ball leaves the spring gun, the ball has speed v_0 with respect to the spring gun.



Problem 2(a) *(15 points)*
What is the speed v_f at which the spring gun moves, immediately after the ball leaves the spring gun?

Space for working Problem 2(a)

external forces → momentum not conserved in y direction, conserved in x direction

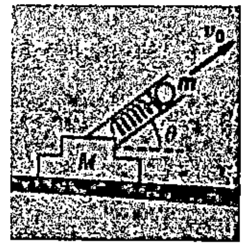
$$MV_f + mV_0\cos\theta = 0$$

$$\boxed{V_f = \frac{-mV_0\cos\theta}{M}}$$

Additional space for working Problem 2(a)

Problem 2(b) starts on the next page→

A spring gun fires a ball at an angle θ with respect to the horizontal. The mass of the ball is m, and the mass of the spring gun is M. The spring gun is initially at rest, and it is on a frictionless surface. At the time the ball leaves the spring gun, the ball has speed v_0 with respect to the spring gun.



Problem 2(b) *(10 points)*

Now, assume that the angle θ = 0. If the spring is originally compressed by a distance x_0 (along the length of the spring), what is the spring gun speed v_f in terms of this distance?

Space for working Problem 2(b)

$$\cancel{1/2}\, k x_0^2 = \cancel{1/2}\, m V_0^2 + \cancel{1/2}\, M V_f^2 \quad , \quad \cancel{V_f(\theta) = -\frac{m V_0 \cos\theta}{M}}$$

$$\cancel{V_f(0) = -\frac{m}{M} V_0}$$

$$k x_0^2 - m V_0^2 = M V_f^2$$

$$\cancel{\boxed{V_f = \sqrt{\frac{k x_0^2 - m V_0^2}{M}}}}$$

$$m V_0 + M V_f = 0$$

$$V_0 = -\frac{M V_f}{m}$$

$$k x_0^2 + \cancel{m}\left(\frac{M^2 V_f^2}{m^{\cancel{2}}}\right) = M V_f^2$$

$$k x_0^2 = M V_f^2 + \frac{M^2 V_f^2}{m}$$

$$k x_0^2 = V_f^2 (M + M^2/m)$$

$$\boxed{V_f = \sqrt{\frac{k x_0^2}{M + \frac{M^2}{m}}}}$$

Additional space for working Problem 2(b)

Problem 3 starts on the next page→

Problem 3 *(4 parts (a)-(d); 45 points + 10 extra credit points)*

An object of mass m falls downwards from height h, starting at rest. It is subject to gravity, and to air resistance, which exerts a force $F=-\xi v$, where v is the velocity of the object, and ξ is a constant.

Problem 3(a) *(25 points)*

Find an expression for y(t), which denotes its position in the vertical direction.

Space for working Problem 3(a)

$$ma = -\xi v - mg \qquad b = \frac{\xi}{m}$$

$$a = -\frac{\xi v}{m} - g$$

$$\frac{dv}{dt} = -bv - g$$

$$\int \frac{dv}{-bv-g} = \int_0^t dt$$

$$\ln(-bv-g) + V_0 = t \quad , \quad V_0 = 0$$

$$-bv - g = e^t$$

$$-bv = e^t + g$$

$$V(t) = \frac{e^t + g}{-b}$$

$$\int dy = \int_0^t \frac{e^t + g}{-b}\, dt$$

$$y(t) + X_0 = -\frac{1}{b}\int_0^t e^t dt - \frac{g}{b}\int_0^t dt$$

$$y(t) = -\frac{1}{b}(e^t - 1) - \frac{gt}{b}$$

$$\boxed{y(t) = -\frac{m}{\xi}(e^t - 1) - \frac{mgt}{\xi}}$$

Additional space for working Problem 3(a)

Problem 3(b) starts on the next page→

An object of mass m falls downwards from height h, starting at rest. It is subject to gravity, and to air resistance, which exerts a force $F=-\xi v$, where v is the velocity of the object, and ξ is a constant.

Additional space for working Problem 3(a)

Problem 3(b) starts on the next page→

Problem 3(b) *(5 points)*

(b) Let t_f be the time at which the object hits the ground. (Note: you do not need to solve for this t_f.) Find v_f, the velocity of the object when it hits the ground, in terms of this t_f. Simplify this expression as much as possible.

Space for working Problem 3(b)

$$V(t) = \frac{e^{t}+g}{-b} = -\frac{m}{\xi}\left(e^{t}+g\right)$$

$$\boxed{V(t_f) = -\frac{m}{\xi}\left(e^{t_f}+g\right)}$$

An object of mass m falls downwards from height h, starting at rest. It is subject to gravity, and to air resistance, which exerts a force $F=-\xi v$, where v is the velocity of the object, and ξ is a constant.

Problem 3(c) *(15 points)*

Find the energy that the object dissipates due to air resistance, as it drops from height h to 0. Do this calculation by finding the kinetic and potential energies at starting and final conditions. Note: the answer will be in terms of t_f.

Space for working Problem 3(c)

$$E_i = mgh$$

$$E_f = \tfrac{1}{2} m \left(\frac{m^2}{\xi^2} \left(e^{t_f} + g\right)^2 \right)$$

$$\boxed{\text{energy dissipated} = mgh - \frac{1}{2}\frac{m^3}{\xi^2}\left(e^{t_f} + g\right)^2}$$

Problem 3(d) *(Extra credit, 10 points)*

Note: you do not need to do this part, unless you have extra time during the exam. Now, find the energy that the object dissipates due to air resistance, by a direct calculation.

Space for working Problem 3(d)

Additional space for working Problem 3(d)