Final Exam
Physics 1AH, Fall 2021
Friday, December 10, 11:30 am

Exam Format
- There are six problems on this exam, each with multiple parts. The point values for each part of each problem are indicated on the exam.
- You will have 3 hours to complete this exam.
- Please show your work on each problem.
- Unless stated otherwise, write all answers in terms of the symbols given in the problem and the physical constant $g \approx +9.8\ \mathrm{m/s^2}$.

Completing Your Exam
- Write your name and UID on this page.
- This exam has **32** pages. Once the exam starts, please check your exam to make sure all pages are included.
- Each part of each problem appears at the top of a page. This exam is printed double-sided, so check both the front and back of each page.
- Write all answers legibly inside the designated boxes.
- This exam is printed with ample space for scratch work. If you need extra scratch paper, raise your hand (do not leave your seat) and a proctor will bring some. Write your name, UID, and the problem and part number on each piece of scratch paper you use.

Resources
- You may reference your optional two double-sided 8.5-by-11 sheets of notes during the exam. You may not use any other resources (including print and electronic resources) during the exam. Calculators are not allowed.
- You may not communicate with anyone except the course staff during the exam.
- Some students will take this exam late. After completing the exam, do not discuss the exam with anyone who has not yet taken it.

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Problem 1 (Three parts (a)-(c); 20 points)

Problem 1(a) (10 points)

A simple harmonic oscillator (mass m, connected to a spring of stiffness k) is subject to a friction force of $-bv$, where v is the velocity of the mass and b is a positive constant. Starting with the differential equation, derive the position of the oscillator as a function of time, $x(t)$.

Assume that the system is overdamped, so $\frac{\gamma}{2} > \omega_0$. Further, assume that the mass starts from rest, at displacement X_0.

Please show your work. If your solution is in terms of variables other than m, k, b, and X_0, please define those variables.

Space for working Problem 1(a)

$F = -kx - bv$ $\qquad \gamma = b/m$

$x = A e^{-\gamma t/2} \cos(\omega_1 t + \phi)$, A = amplitude

$= e^{-\gamma t/2}\left[B\cos(\omega_1 t) + C\sin(\omega_1 t)\right]$, $A = \sqrt{B^2 + C^2}$

$x(0) = X_0 = B\cos(0) + C\sin(0) = B \rightarrow X_0 = B$

$v(t) = -A e^{-\gamma t/2}\left[\omega_1 \sin(\omega_1 t + \phi) + \gamma/2 \cos(\omega_1 t + \phi)\right]$

$v(0) = 0 = -A\left[\omega_1 \sin(\phi) + \gamma/2 \cos(\phi)\right]$

$\omega_1 \sin(\phi) + \gamma/2 \cos(\phi) = 0$, $\phi = \tan^{-1}(B/C)$

Additional space for working Problem 1(a)

$$m\ddot{x} = -kx - b\dot{x}$$

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$$

$$\frac{dv}{dt} = -\frac{k}{m}x - \frac{b}{m}v$$

Problem 1(b) starts on the next page→

A simple harmonic oscillator (mass m, connected to a spring of stiffness k) is subject to a friction force of $-bv$, where v is the velocity of the mass and b is a positive constant. Starting with the differential equation, derive the position of the oscillator as a function of time, $x(t)$.

Assume that the system is overdamped, so $\frac{\gamma}{2} > \omega_0$. Further, assume that the mass starts from rest, at displacement X_0.

Please show your work. If your solution is in terms of variables other than m, k, b, and X_0, please define those variables.

Additional space for working Problem 1(a)

Problem 1(b) *(5 points)*

Now, assume that the damping is strong ($\omega_0 \ll \gamma$). Use the Taylor series expansion to find the leading term (first nonzero term) in your expression for $x(t)$.

Space for working Problem 1(b)

$$x(t) \approx x_0 + \frac{dx}{dt}(t - t_0) + \frac{1}{2} \cdot \frac{d^2x}{dt^2}(t - t_0)^2 + \cdots$$

Problem 1(c) starts on the next page→

Problem 1(c) *(5 points)*

Starting from your original differential equation, assume that the inertial term (the $\ddot{x}$ term) can be neglected. Derive the position $x(t)$, again assuming that the object starts from rest at X_0. Check how your answer compares to that found in part (b).

Space for working Problem 1(c)

$m\ddot{x} = -kx - b\dot{x}$

$0 = -kx - b\dot{x}$

$b\dot{x} = -kx$

$b\frac{dx}{dt} = -kx$

$\frac{dx}{dt} = \frac{-k}{b}x$

$\frac{1}{x}dx = -\frac{k}{b}dt$

$\int_{x_0}^{x(t)} \frac{1}{x}dx = \int_0^t -\frac{k}{b}dt$

$\ln(x(t)) - \ln(x_0) = -\frac{k}{b}t$

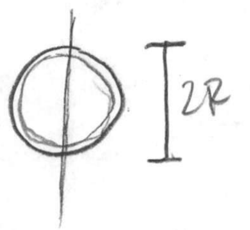
$\boxed{x(t) = e^{-kt/b} + X_0}$

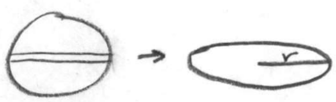
Problem 2 *(Four parts (a)-(d); 20 points + 5 extra credit points)*

Problem 2(a) *(10 points)*

Calculate the moment of inertia for a hollow spherical shell of mass M and radius R. Assume the axis around which the moment of inertia is computed goes through the center of the sphere.

Space for working Problem 2(a)





$$I = \int r^2 dm$$

$$\rho = \frac{m}{A} \rightarrow \rho = \frac{m}{2\pi r}$$

$$2\pi r \rho = m$$

$$2\pi \rho r dr = dm$$

$$\rho = \frac{M}{4\pi R^2}$$

$$I = 2\int_0^R r^3 (2\pi\rho)\, dr$$

$$= 4\pi\rho \int_0^R r^3 dr$$

$$= \frac{\cancel{4}\pi\rho R^4}{\cancel{4}}$$

$$= \pi R^{\cancel{4}2} \cdot \frac{M}{4\cancel{\pi}\cancel{R^2}} = \boxed{\frac{1}{4} M R^2}$$

SA of sphere $= 4\pi R^2$

Problem 2(b) starts on the page after next→

Calculate the moment of inertia for a hollow spherical shell of mass M and radius R. Assume the axis around which the moment of inertia is computed goes through the center of the sphere.

Additional space for working Problem 2(a)

Additional space for working Problem 2(a)

Problem 2(b) starts on the next page→

Problem 2(b) *(5 points)*

Calculate the moment of inertia for a solid sphere, mass M, radius R. As in the previous case, assume that the axis goes through the center of the sphere.

Space for working Problem 2(b)

take many hollow spheres

$$\rho = \frac{m}{4\pi r^2} \rightarrow dm = 4\rho\pi r^2 dr$$

$$\rho = \frac{M}{4/3\pi R^3} \rightarrow dm = 4\pi r^2 \cdot \frac{3M}{4\pi R^3} dr$$

$$= r^2 \cdot \frac{3M}{R^3} dr$$

$$I = \int r^2 dm$$

$$I = \int_0^R r^4 \left(\frac{3M}{R^3}\right) dr$$

$$= \frac{3M}{R^3}\left[\frac{r^5}{5}\right]_0^R$$

$$= \frac{3MR^{\cancel{5}2}}{5\cancel{R^3}} = \boxed{\frac{3}{5}MR^2}$$

Volume of a sphere: $4/3\,\pi R^3$

Additional space for working Problem 2(b)

Problem 2(c) starts on the next page→

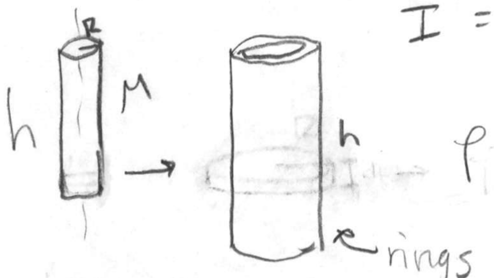
Calculate the moment of inertia for a solid sphere, mass M, radius R. As in the previous case, assume that the axis goes through the center of the sphere.

Additional space for working Problem 2(b)

Problem 2(c) (5 points)

What is the moment of inertia of a solid cylinder, radius R, height h, and mass M? Assume that the z-axis lies along the axis of rotational symmetry of the cylinder, i.e. the length of the cylinder is along z, and the cross-section in xy-plane is a circle. Compute the moment of inertia around the z-axis.

Space for working Problem 2(c)

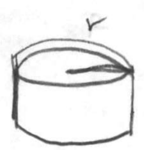


$$I = \int r^2 \, dm$$

$$\rho = \frac{m}{2\pi r h} \rightarrow dm = 2\pi h r \rho \, dr$$

$$\rho = \frac{M}{\pi R^2 h}$$

moment of inertia for a disk is the same as moment of inertia for a cylinder: take rings.



moment of inertia for a ring is mr^2

$$dm = 2\pi h r \cdot \frac{M}{\pi R^2 h} dr = \frac{2Mr}{R^2} dr$$

$$I = \int_0^R r^2 \cdot \frac{2Mr}{R^2} dr$$

$$= \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R = \frac{2MR^4}{4R^2} = \boxed{\frac{MR^2}{2}}$$

Problem 2(d) starts on the next page→

What is the moment of inertia of a solid cylinder, radius R, height h, and mass M? Assume that the z-axis lies along the axis of rotational symmetry of the cylinder, i.e. the length of the cylinder is along z, and the cross-section in xy-plane is a circle. Compute the moment of inertia around the z-axis.

Additional space for working Problem 2(c)

Problem 2(d) *(Extra credit, 5 points)*
Note: you do not need to do this part, unless you have extra time during the exam.

Find the moment of inertia of a solid cylinder around the y-axis. Assume that the origin is at the center of the cylinder. As described above, the cross section of the cylinder in the xy-plane is a circle.

Space for working Problem 2(d)

Problem 3 starts on the page after next→

Find the moment of inertia of a solid cylinder around the y-axis. Assume that the origin is at the center of the cylinder. As described above, the cross section of the cylinder in the xy-plane is a circle.

Additional space for working Problem 2(d)

Additional space for working Problem 2(d)

Problem 3 starts on the next page→

Problem 3 *(3 parts (a)-(c); 10 points)*

A pendulum consisting of a mass m attached at the end of a string of length l is released from rest, at angle of θ_0 from the vertical axis. The string itself is massless.

Problem 3(a) *(5 points)*

Assuming that the initial angle is small, find the differential equation that describes the motion of the pendulum. Keep only the first term in your Taylor series expansion; this term will be linear in θ. Then solve the equation to find the angle as a function of time. Please show your work.

Space for working Problem 3(a)

$$\theta(t) = \theta_0 + \frac{d\theta}{dt}(t - t_0) + \frac{d^2\theta}{dt^2}(t - t_0)^2 + \dots$$

$$mg = T\cos\theta$$

$$T = \frac{mg}{\cos\theta}$$

$$T = \frac{mv^2}{l}$$

$$\frac{mg}{\cos\theta} = \frac{mv^2}{l}$$

$$\frac{g}{\cos\theta} = \omega^2 l$$

$$= \dot{\theta}^2 l$$

$$\dot{\theta} = \frac{g}{l\cos\theta}$$

$$\frac{d\theta}{dt} = \frac{g}{l\cos\theta}$$

$$\cos\theta \, d\theta = \frac{g}{l} dt$$

$$\int \cos\theta \, d\theta = \int \frac{g}{l} dt$$

$$\sin\theta + \theta_0 = \frac{gt}{l}$$

$$\frac{d\theta}{dt} = \frac{g}{l\cos\theta}$$

$$\frac{d^2\theta}{dt^2} = \frac{g}{l} \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{g}{l} \tan\theta \sec\theta$$

Additional space for working Problem 3(a)

$$\theta(t) = \theta_0 + \frac{g}{\ell\cos\theta} t + \frac{g}{\ell}\tan\theta\sec\theta\, t^2$$

Problem 3(b) starts on the next page→

A pendulum consisting of a mass m attached at the end of a string of length l is released from rest, at angle of θ_0 from the vertical axis. The string itself is massless.

Problem 3(b) *(3 points)*

Now, assume that the angle is slightly larger, so that the Taylor series expansion has to be carried to the next highest term. Find the differential equation that describes the motion of the pendulum. Note: you do not need to solve it; only show the equation.

Space for working Problem 3(b)

Problem 3(c) *(2 points)*

Check that solutions which normally describe simple harmonic motion are no longer valid.

Space for working Problem 3(c)

Problem 4 starts on the next page→

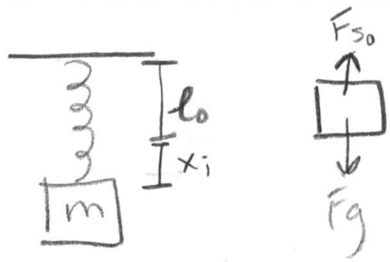
Problem 4 *(Three parts (a)-(c); 20 points)*

A mass m is hung from an elastic spring of stiffness k and equilibrium length l_0. The mass swings back and forth, such that the pendulum makes an angle $\theta(t)$ with respect to the vertical axis. Assume that the radial component of the net force on the mass is always negligible.

Problem 4(a) *(5 points)*

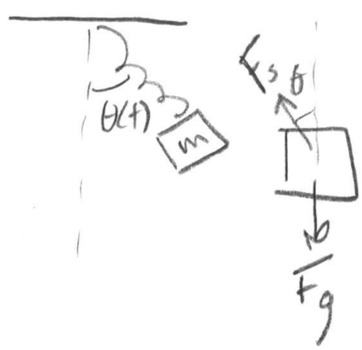
Find the length of the spring as a function of the angle, $l(\theta)$.

Space for working Problem 4(a)



$$F_{s_0} = mg = kx_i$$

$$l(\theta) = l_0 + \Delta l$$



$$mg = F_s \cos(\theta(t))$$

$$F_s = k\Delta l$$

$$mg = k(l(\theta) - l_0)\cos(\theta(t))$$

$$\frac{mg}{k\cos(\theta(t))} = l(\theta) - l_0$$

$$\boxed{l(\theta) = \frac{mg}{k\cos(\theta)} + l_0}$$

Problem 4(b) *(10 points)*

Find the differential equation for $\theta(t)$. You do not need to solve it.

Space for working Problem 4(b)

Problem 4(c) starts on the next page→

A mass m is hung from an elastic spring of stiffness k and equilibrium length l_0. The mass swings back and forth, such that the pendulum makes an angle $\theta(t)$ with respect to the vertical axis. Assume that the radial component of the net force on the mass is always negligible.

Problem 4(c) (5 points)

Now assume the small angle approximation: $\theta \ll 1$ at all times. You may therefore expand the expression up to first order in θ. Find the frequency of oscillation.

Space for working Problem 4(c)

Problem 5 *(Two parts (a)-(b); 20 points)*

A simple harmonic oscillator (mass m connected to a spring of stiffness k) moving on a surface with friction (which exerts a force $-fv$, where v is the velocity of the mass and ~~k~~ f is a positive constant) is driven with a force $F_0 \cos \omega t$.

Problem 5(a) *(15 points)*

Derive the position of the mass as a function of time, making no assumptions about the relative values of the different parameters of the system. You may neglect any transients.

Please show your work. If your solution is in terms of variables other than $m, k, f,$ F_0, and ω, please define those variables.

Space for working Problem 5(a)

$$m\ddot{x} = -f\dot{x} - kx + F_0 \cos\omega t$$

$$\ddot{x} = -\frac{f}{m}\dot{x} - \frac{k}{m} + \frac{F_0}{m}\cos\omega t \quad , \quad \text{let } \gamma = \frac{f}{m}, \ \omega_0 = \sqrt{\frac{k}{m}}$$

$$\ddot{x} = -\gamma\dot{x} - \omega_0^2 + \frac{F_0}{m}\cos\omega t$$

$$\frac{dv}{dt} = -\gamma v - \omega_0^2 + \frac{F_0}{m}\cos\omega t$$

~~$v(t) =$~~

$$v(t) = \underbrace{-\omega\frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)+(\omega\gamma)^2}}}_{\text{constant}} \sin(\omega t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{\gamma/2}{\omega_0 - \omega}\right)$$

Problem 5(b) starts on the page after next→

A simple harmonic oscillator (mass m connected to a spring of stiffness k) moving on a surface with friction (which exerts a force $-fv$, where v is the velocity of the mass and b is a positive constant) is driven with a force $F_0 \cos \omega t$.

Derive the position of the mass as a function of time, making no assumptions about the relative values of the different parameters of the system. You may neglect any transients.

Please show your work. If your solution is in terms of variables other than m, k, f, F_0, and ω, please define those variables.

Additional space for working Problem 5(a)

$$x(t) = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2-\omega^2)+(\omega\gamma)^2}} \cos(\omega t + \phi)$$

Additional space for working Problem 5(a)

Problem 5(b) starts on the next page→

Problem 5(b) *(5 points)*

Now, assume that the system is driven exactly at resonance, $\omega = \omega_0$. Find the position $x(t)$. What do you notice about the phase of this motion with respect to the drive?

Space for working Problem 5(b)

$$x(t) = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2-\omega^2)+(\omega\gamma)^2}} \cos(\omega t + \phi) \quad , \quad \phi = \tan^{-1}\left(\frac{\gamma/2}{\omega_0-\omega}\right)$$

when $\omega = \omega_0$

$$x(t) = \frac{F_0}{m} \frac{1}{\omega_0 \gamma} \cos(\omega_0 t + \phi)$$

where $\phi = \tan^{-1}\left(\frac{\gamma/2}{\omega_0-\omega}\right)$, let $\frac{\gamma/2}{\omega_0-\omega} = b$

as $\omega \rightarrow \omega_0$, $b \rightarrow \infty$

as $b \rightarrow \infty$, $\tan^{-1}(b) \rightarrow \pi/2$

but, when $\omega = \omega_0$ exactly, the phase of the motion is undefined

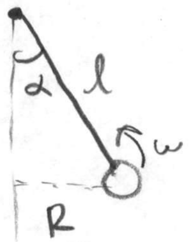
Problem 6 *(Three parts (a)-(c); 10 points)*

A pendulum consists of a mass m attached at the end of a massless string of length l. The mass is rotating around the z-axis in a circle, at constant angular speed ω. The string is observed to make a constant angle α with the z-axis. Assume that the origin is at the top, where the pendulum is attached.

Problem 6(a) *(5 points)*

Find the total torque on the mass m about the origin.

Space for working Problem 6(a)



$\tau = F \cdot r$

$\cos\alpha = mg/T$

$T\cos\alpha = mg$

$T = \dfrac{mg}{\cos\alpha}$

$\sin\alpha = T_x/T$

$T_x = T\sin\alpha$

$= mg\tan\alpha$

$R = l\sin\alpha$

$\cancel{F = ma}$

$\cancel{mg\tan\alpha = m\,v^2/R, \quad v = \omega R}$

$\cancel{g\tan\alpha}$

$\tau = I\alpha, \quad \alpha = a/R$

$I = mR^2$

$a = \omega^2 R$

$\tau = mR^2\omega^2$

$\boxed{\tau = m\omega^2 l^2 \sin^2\alpha}$

$\tau = T_x \cdot R$

$= mg\tan\alpha\,(l\sin\alpha)$

$\boxed{\tau = mg\,l\tan\alpha\sin\alpha}$

Problem 6(b) starts on the next page→

A pendulum consists of a mass m attached at the end of a massless string of length l. The mass is rotating around the z-axis in a circle, at constant angular speed ω. The string is observed to make a constant angle α with the z-axis. Assume that the origin is at the top, where the pendulum is attached.

Problem 6(b) *(3 points)*

Find the angular momentum of the mass m about the origin.

Space for working Problem 6(b)

$L = mvR, \quad v = \omega R$

$L = m\omega R^2, \quad R = \ell \sin\alpha$

$\boxed{L = m\omega \ell^2 \sin^2\alpha}$

Problem 6(c) *(3 points)*

Demonstrate that $\vec{\tau} = \frac{d\vec{L}}{dt}$.

Space for working Problem 6(c)

$\tau = m\omega^2 l^2 \sin^2\alpha$, $L = m\omega l^2 \sin^2\alpha$

$\frac{dL}{dt}$

Additional space for working Problem 6(c)