Quiz 3 Solutions

Question 1. (5 points) Find the maxima and minima of the function

$$f(x,y,z) = 3x^2 + y^2 + z^2$$

on the surface $x^2 + y^2 + 3z^2 = 3$.

The surface is an ellipsoid, which is closed and bounded. Also, f is continuous. Therefore there is a point (x_m, y_m, z_m) attaining the minimum and a point (x_M, y_M, z_M) attaining the maximum.

The constraint equation is $x^2+y^2+3z^2 = 3$. The functions $x^2+y^2+3z^2$ and f are totally differentiable everywhere. By the theorem of Lagrange multipliers, at each of (x_m, y_m, z_m) and (x_M, y_M, z_M), either the gradient of $x^2+y^2+3z^2$ is 0, or the gradient of f is a (possibly 0) scalar multiple of $x^2+y^2+3z^2$.

The gradient of $x^2+y^2+3z^2$ at any point (x_0, y_0, z_0) is $\langle 2x_0, 2y_0, 6z_0\rangle$ which is only 0 at $(x_0, y_0, z_0) = (0,0,0)$, but this is not on the surface $x^2+y^2+3z^2 = 3$ so it is impossible for the gradient of $x^2+y^2+3z^2$ to be 0 at either (x_m, y_m, z_m) or (x_M, y_M, z_M).

Therefore, at each of (x_m, y_m, z_m) and (x_M, y_M, z_M), the gradient of f is a (possibly 0) scalar multiple of $x^2+y^2+3z^2$.

The gradient of f at any point (x_0, y_0, z_0) is $\langle 6x_0, 2y_0, 2z_0\rangle$.

We get that there exists some $\lambda \in \mathbf{R}$ such that

$$6x_m = \lambda 2x_m$$

$$2y_m = \lambda 2y_m$$

$$2z_m = \lambda 6z_m$$

$$x_m^2 + y_m^2 + 3z_m^2 = 3$$

and the same equations with each of x_m, y_m, z_m replaced with x_M, y_M, z_M, respectively.

We can rearrange each of the first 3 equations to get

$$(6-2\lambda)x_m = 0$$

$$(2-2\lambda)y_m = 0$$

$$(2-6\lambda)z_m = 0.$$

At most one of $6-2\lambda, 2-2\lambda, 2-6\lambda$ can be 0 (because no real value of λ will make more than one of them 0), so at least two of x_m, y_m, z_m are 0.

Combining this with $x_m^2 + y_m^2 + 3z_m^2 = 3$ gives that (x_m, y_m, z_m) must be one of the points $(0,0,1), (0,0,-1), (0,\sqrt{3},0), (0,-\sqrt{3},0), (\sqrt{3},0,0), (-\sqrt{3},0,0)$.

The same analysis shows that (x_M, y_M, z_M) must also be one of those points.

We conclude that (at least) one of the points $(0,0,1), (0,0,-1), (0,\sqrt{3},0), (0,-\sqrt{3},0), (\sqrt{3},0,0), (-\sqrt{3},0,0)$ minimizes f over the surface $x^2+y^2+3z^2 = 3$, and (at least) one of them maximizes f over this surface.

To find out which one(s) minimize/maximize f, we evaluate at each point:

(x,y,z)	$f(x,y,z)$
$(0,0,1)$	1
$(0,0,-1)$	1
$(0,\sqrt{3},0)$	3
$(0,-\sqrt{3},0)$	3
$(\sqrt{3},0,0)$	9
$(-\sqrt{3},0,0)$	9

The last 4 points do not minimize f over the surface $x^2+y^2+3z^2 = 3$, because a lower value of f can be attained at $(0,0,\pm 1)$. We conclude that (x_m, y_m, z_m) must be one of $(0,0,\pm 1)$, and hence $f(x_m, y_m, z_m) = 1$. Therefore, the minimum of f over the surface $x^2+y^2+3z^2 = 3$ is 1, attained at $(0,0,\pm 1)$.

Similarly, the maximum is 9, attained at $(\pm\sqrt{3},0,0)$.

Question 2. Let a and b be positive real numbers such that $a < b$.

(a) (3 points) Determine the global maxima and minima of the function

$$f(x, y) = (ax^2 + by^2)e^{-x^2-y^2}$$

whose domain is the whole plane $\mathbb{R}^2$.

(b) (2 points) Find the maxima and minima of $f(x, y)$ on the unit circle $x^2 + y^2 = 1$.

(a) Solution 1: To find the minimum, both factors are nonnegative, so $f(x, y) \geq 0$ for all $(x, y) \in \mathbf{R}^2$; furthermore, the second factor $e^{-x^2-y^2}$ is never 0, and the first factor $(ax^2 + by^2)$ is 0 only at the origin. So $f(0, 0) = 0 = \min_{(x,y)\in\mathbf{R}^2} f(x, y)$, and $f(x, y) > 0 = f(0, 0)$ for any point $(x, y) \neq (0, 0)$, so the minimum is 0 which is attained only at $(0, 0)$.

To find the maximum, we first reduce the problem to a maximization problem over a closed, bounded region. For any $r > 0$, the closed disk $x^2 + y^2 \leq r^2$ is closed and bounded.

Because exponentials grow faster than polynomials, for each $\epsilon > 0$, there exists $r_\epsilon > 0$ such that $ax^2 + by^2 \leq \epsilon e^{x^2+y^2}$ for all (x, y) with $x^2 + y^2 \geq r_\epsilon^2$ (indeed, $ax^2 + by^2 \leq a(x^2 + y^2) + b(x^2 + y^2) = (a + b)(x^2 + y^2)$ which is less than $\epsilon e^{x^2+y^2}$ for large values of $x^2 + y^2$). Then $f(x, y) \leq \epsilon$ for (x, y) satisfying $x^2 + y^2 \geq r^2$.

Taking $\epsilon_0 = f(1, 1)/2$, we get that, if (x_M, y_M) is a maximizer of f over the closed disk $x^2 + y^2 \leq r_{\epsilon_0}^2$ (which exists by the extreme value theorem, because this closed disk is closed and bounded and f is continuous on this disk), then for any point (x, y) outside this disk, $f(x_M, y_M) \geq f(1, 1) > \epsilon_0 > f(x, y)$. (Here, we use that $(1, 1)$ must be in the disk, because $f(1, 1) > \epsilon_0$.)

We conclude that for any point $(x, y) \in \mathbf{R}^2$, $f(x_M, y_M) \geq f(x, y)$, so a maximizer of f over the closed disk $x^2 + y^2 \leq r_{\epsilon_0}^2$ is a global maximizer of f.

Conversely, if (x_M, y_M) is any global maximizer of f, then $f(x_M, y_M) \geq f(1, 1) > \epsilon_0$, and hence (x_M, y_M) must be in the disk $x^2 + y^2 \leq r_{\epsilon_0}^2$, and therefore must be a maximizer of f over the disk.

We conclude that a point (x, y) is a global maximizer of f if and only if (x, y) is a maximizer of f over the disk $x^2 + y^2 \leq r_{\epsilon_0}^2$.

Next, we show that any maximizer of f over the disk must be in the interior of the disk. Indeed, for any point (x, y) on the boundary, $x^2 + y^2 = r_{\epsilon_0}^2$, so $f(x, y) \leq \epsilon = f(1, 1)/2 < f(1, 1)$, so (x, y) cannot be maximizer of f over the disk because $(1, 1)$ is a point in the disk attaining a larger value.

We have shown that any maximizer (x_M, y_M) of f over the disk must be in the interior of the disk. Because f has partial derivatives everywhere in the disk (in fact, it is totally differentiable everywhere), $f_x(x_M, y_M) = f_y(x_M, y_M) = 0$ (where subscripts denote partial derivatives). Expanding,

$$2ax_M e^{-x_M^2-y_M^2} + (ax_M^2 + by_M^2)(-2)x_M e^{-x_M^2-y_M^2} = 0$$

$$2by_M e^{-x_M^2-y_M^2} + (ax_M^2 + by_M^2)(-2)y_M e^{-x_M^2-y_M^2} = 0.$$

Factoring,

$$2e^{-x_M^2-y_M^2} x_M(a - ax_M^2 - by_M^2) = 0$$

$$2e^{-x_M^2-y_M^2} y_M(b - ax_M^2 - by_M^2) = 0.$$

From the first equation, either $x_M = 0$ or $a - ax_M^2 - by_M^2 = 0$ (or both). From the second equation, either $y_M = 0$ or $b - ax_M^2 - by_M^2 = 0$ (or both).

We get the following 4 possibilities, at least one of which must occur:

- $x_M = 0$ and $y_M = 0$: In this case $(x_M, y_M) = (0,0)$ (I will defer ruling this out until after examining all cases).
- $x_M = 0$ and $b - ax_M^2 - by_M^2 = 0$: In this case $(x_M, y_M) = (0, \pm 1)$.
- $a - ax_M^2 - by_M^2 = 0$ and $y_M = 0$: In this case $(x_M, y_M) = (\pm 1, 0)$.
- $a - ax_M^2 - by_M^2 = 0$ and $b - ax_M^2 - by_M^2 = 0$: Impossible, because the first equation implies $ax_M^2 + by_M^2 = a$, and the second implies $ax_M^2 + by_M^2 = b$, but $a < b$.

So (x_M, y_M) must be one of $(0,0), (0,\pm 1), (\pm 1, 0)$.
Evaluating at each point:

(x,y)	$f(x,y)$
$(0,0)$	0
$(0,\pm 1)$	be^{-1}
$(\pm 1,0)$	ae^{-1}

and as $a < b$, the largest of these is be^{-1}, so the maximum is be^{-1} attained at $(0, \pm 1)$.

Solution 2: on any circle given by an equation of the form $x^2 + y^2 = r^2$ (allowing the degenerate circle when $r = 0$), we can write f as

$$f(x,y) = (ar^2 + (b-a)y^2)e^{-r^2}$$

and the maximum/minimum over the circle could be determined using the theorem of Lagrange multipliers, but a faster way is to observe that the restriction of f to this circle is maximized/minimized precisely when y^2 is, which occurs at $(0, \pm r)$. We can compute the maximum value over the circle to be $br^2e^{-r^2}$.
Similarly, the set of points that minimize y^2 over the circle (and hence that minimize f over the circle) are $(\pm r, 0)$, and the minimum value is $ar^2e^{-r^2}$.
For each $r \geq 0$, let $(x_{r,M}, y_{r,M})$ denote any maximizer of f over the circle $x^2 + y^2 = r^2$ (which exists because we just showed that $(0, r)$ is a maximizer).
A point (x, y) is a global maximizer for f if and only if $f(x,y) \geq f(x_{r,M}, y_{r,M})$ for all $r \geq 0$ (indeed, by definition, (x,y) is a global maximizer if and only if $f(x,y) \geq f(x_0, y_0)$ for all $(x_0, y_0) \in \mathbf{R}^2$, so if (x,y) is a global maximizer then $f(x,y) \geq f(x_{r,M}, y_{r,M})$ for all $r \geq 0$, and conversely if $f(x,y) \geq f(x_{r,M}, y_{r,M})$ for all $r \geq 0$ then for any $(x_0, y_0) \in \mathbf{R}^2$, $f(x,y) \geq f(x_{\sqrt{x_0^2+y_0^2},M}, y_{\sqrt{x_0^2+y_0^2},M}) \geq f(x_0, y_0)$ where the latter inequality is because (x_0, y_0) is on the circle $x^2 + y^2 = \sqrt{x_0^2 + y_0^2}^2$).
Furthermore, if (x_M, y_M) is a global maximizer of f, then so is $(x_{\sqrt{x_M^2+y_M^2},M}, y_{\sqrt{x_M^2+y_M^2},M})$, because $f(x_{\sqrt{x_M^2+y_M^2},M}, y_{\sqrt{x_M^2+y_M^2},M}) \geq f(x_M, y_M)$ (because (x_M, y_M) is on the circle $x^2 + y^2 = \sqrt{x_M^2 + y_M^2}^2$).
Using our computation we did above of the maximum of f over each circle $x^2 + y^2 = r^2$, a point (x,y) is a global maximizer for f if and only if $f(x,y) \geq f(x_{r,M}, y_{r,M}) = br^2e^{-r^2}$ for all $r \geq 0$. By single variable calculus, $br^2e^{-r^2}$ is maximized at $r = 1$ (indeed, $\frac{d}{dr}br^2e^{-r^2} = (2br - br^2 2r)e^{-r^2} = 2bre^{-r^2}(1 - r^2)$, which is positive for $0 < r < 1$ and negative for $1 < r$, so $br^2e^{-r^2}$ increases from $r = 0$ to $r = 1$ and then decreases after that). So a point (x,y) is a global maximizer for f if and only if $f(x,y) \geq f(x_{1,M}, y_{1,M})$, and furthermore $f(x_{r,M}, y_{r,M})$ attains a value at least as large as $f(x_{1,M}, y_{1,M})$ only at $r = 1$.
Combining the previous 2 paragraphs, if (x,y) is a global maximizer, then $x^2 + y^2 = 1$, and a point (x,y) is a global maximizer if and only if it maximizes f over the unit circle. Using our computation of the maximizers of f over the unit circle, the global maximum of f is $f(x_{1,M}, y_{1,M}) = be^{-1}$ attained only at $(0, \pm 1)$.
Similarly, letting $(x_{r,m}, y_{r,m})$ denote any minimizer of f over the circle $x^2 + y^2 = r^2$, a point (x,y) is a global minimizer for f if and only if $f(x,y) \leq f(x_{r,m}, y_{r,m}) = ar^2e^{-r^2}$ for all $r \geq 0$. The right hand side is ≥ 0, with equality if and only if $r = 0$. By more similar arguments, the global minimum of f is $f(x_{0,m}, y_{0,m}) = 0$ attained only at $(0,0)$.

(b) In the previous part of this problem, we showed that $(0, \pm 1)$ attain the maximum value of f over all of $\mathbf{R}^2$. Therefore, $f(0, \pm 1)$ must be at least as large as $f(x, y)$ for any (x, y) on the unit circle.

As $(0, \pm 1)$ are on the unit circle, this implies $(0, \pm 1)$ maximize f over the unit circle.

Also, in the previous part of this problem, we showed that $(0, \pm 1)$ are the only points that attain the maximum value of f over $\mathbf{R}^2$, so in particular no other points on the unit circle can attain the value of $f(0, \pm 1)$.

Therefore, the maximum value of f over the unit circle is $f(0, \pm 1) = b$, attained at $(0, \pm 1)$.

For the minimum: on the unit circle, the factor $e^{-x^2-y^2}$ is always e^{-1}, so it is equivalent to minimize $(ax^2 + by^2)e^{-1}$ over the unit circle.

Solution 1: $(ax^2 + by^2)e^{-1}$ is continuous, and the unit circle is closed and bounded, so a minimizer (x_m, y_m) exists.

Both $(ax^2 + by^2)e^{-1}$ and $x^2 + y^2$ are totally differentiable on $\mathbf{R}^2$, and the gradient of $x^2 + y^2$ is $\langle 2x, 2y \rangle$ which is never 0 on the unit circle, so by the theorem of Lagrange multipliers, there is some $\lambda \in \mathbf{R}$ such that

$$\frac{\partial}{\partial x}(ax^2 + by^2)e^{-1} \restriction_{x=x_m} = 2ax_m e^{-1} = \lambda 2x_m$$

$$\frac{\partial}{\partial y}(ax^2 + by^2)e^{-1} \restriction_{y=y_m} = 2by_m e^{-1} = \lambda 2y_m$$

$$x_m^2 + y_m^2 = 1.$$

Factoring the first two equations,

$$2x_m(ae^{-1} - \lambda) = 0$$

$$2y_m(be^{-1} - \lambda) = 0.$$

Because $a < b$, it is impossible for both $ae^{-1} - \lambda = 0$ and $be^{-1} - \lambda = 0$, so at least one of x_m, y_m must be 0. The only points on the unit circle with at least one coordinate 0 are $(0, \pm 1)$ and $(\pm 1, 0)$. Evaluating, $f(0, \pm 1) = be^{-1}$ and $f(\pm 1, 0) = ae^{-1}$, and using $a < b$, this implies the minimum of f over the unit circle is ae^{-1}, attained only at $(\pm 1, 0)$.

Solution 2: (Essentially, doing the computation we did in Solution 2 for part (a) of ths problem.) $(ax^2 + by^2)e^{-1} = (a(x^2 + y^2) + (b - a)y^2)e^{-1}$, and again using $x^2 + y^2 = 1$ on the domain of interest, $f(x, y) = (a + (b - a)y^2)e^{-1}$ for all points (x, y) on the unit circle $x^2 + y^2 = 1$.

Since this is linear in y^2 with positive slope, this is minimized when y^2 is minimized, which occurs precisely at $(\pm 1, 0)$.

The value of f at these points can be computed to be ae^{-1}.