Math 32A - Fall 2021
Midterm 2, Wednesday Nov 10

Full Name: ______________________________

UID: ______________________________

Circle the name of your TA and the day of your discussion:

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Tuesday Thursday

Instructions:

- Write your name and UID on each page.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Write your answers in the space provided.
- Calculators are not permitted.

Page	Points	Score
1	20	
2	20	
3	20	
Total:	60	

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You may use this page for scratch work. Work found on this page will not be graded.

Name: UID:

1. (20 points) Let $f(x, y)$ be the function $x^2 + y$.

(a) Draw the level curves of f in the xy plane for levels $0, -1, 1, 3$. Describe how the level curve changes as the level changes.

Solution: The level curves are all parabolas $y = a - x^2$, where a is the level. As the level changes, the intersection of the level with the y-axis changes, as the level is the y-intercept of the curve. The curves stay parallel to each other.

(b) Parameterize the level curve of f corresponding to level 2 and compute the unit tangent vector and unit normal vector for the curve at $(1, 1)$.

Solution: The level curve of f at level 2 has equation $x^2 + y = 2$. y is hence just a function of x and so we can use x as the parameter itself. So, we get a parameterization

$$\mathbf{r}(t) = \langle t, 2 - t^2 \rangle.$$

The tangent vector at $(1, 1)$ is

$$\mathbf{r}'(t) = \langle 1, -2t \rangle = \langle 1, -2 \rangle.$$

Hence, the unit tangent vector is $\frac{1}{\sqrt{5}}\langle 1, -2 \rangle$. The unit normal vector at this point has to be perpendicular to the unit tangent vector. Hence, we have two possible choices

$$\frac{1}{\sqrt{5}}\langle 2, 1 \rangle \text{ or } \frac{1}{\sqrt{5}}\langle -2, -1 \rangle.$$

To determined which one is correct, note that the level curve is concave down, so the normal vector at any points downwards. So, the normal vector must be the one with both components negative.

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2. (20 points) Let $f(x,y)$ be a function of two variables and let P be a point in the xy-plane.

   (a) Suppose $\lim_{(x,y)\to P} f(x,y) = L$. Define a new function $g(x,y) = f(x,y) + 2f(x,y)^2$. Show that

$$\lim_{(x,y)\to P} g(x,y) = L + 2L^2.$$

   Solution: This follows from the sum and product rules of limits.

   (b) Let f be the function

$$f(x,y) = \frac{x^2 y}{x^2+y^2} + 2\frac{x^4 y^2}{(x^2+y^2)^2}.$$

   Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that it does not exist.

   Solution: If we define $h(x,y) = \frac{x^2 y}{x^2+y^2}$, then $f = h + 2h^2$. Applying the previous part of the problem, we see that if L is the limit of h, if it exists, then f has limit $L + 2L^2$. Now, using polar coordinates

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = \lim_{r\to 0} r\cos^2(\theta)\sin(\theta).$$

   Since $|\cos^2(\theta)\sin(\theta)| \leq 1$, using the squeeze theorem, we see the limit is 0.

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3. (20 points) Let C be the curve with parameterization

$$\mathbf{r}(\theta) = \langle \theta, 1 - \sin(\theta) \rangle.$$

(a) Compute the curvature of C at the point $\langle \frac{\pi}{2}, 1 \rangle$.

Solution: Let us begin by computing the unit tangent vector.

$$\mathbf{r}'(\theta) = \langle 1, -\cos(\theta) \rangle.$$

So the unit tangent vector is

$$\mathbf{T}(\theta) = \frac{1}{\sqrt{1 + \cos^2\theta}} \langle 1, -\cos\theta \rangle.$$

So,

$$\mathbf{T}'(\theta) = \frac{\cos\theta \sin\theta}{(1 + \cos^2\theta)^{\frac{3}{2}}} \langle 1, -\cos\theta \rangle + \frac{1}{\sqrt{1 + \cos^2\theta}} \langle 0, \sin\theta \rangle.$$

The key to making this computation easier is to evaluate now at $\frac{\pi}{2}$, which gives us

$$\mathbf{T}'(\frac{\pi}{2}) = \langle 0, 1 \rangle.$$

Hence, the curvature is

$$\frac{||\mathbf{T}'(\frac{\pi}{2})||}{||\mathbf{r}'(\frac{\pi}{2})||} = 1$$

and the unit normal vector is

$$\frac{\mathbf{T}'(\frac{\pi}{2})}{||\mathbf{T}'(\frac{\pi}{2})||} = \langle 0, 1 \rangle.$$

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(b) Suppose an object moves along C in the same direction as the one determined by the parameterization but with $\mathbf{r}(t)$ a potentially different function than the one described by the parameterization. When the object is at $\langle \frac{\pi}{2}, 1\rangle$, it has speed $2ms^{-1}$ and the speed is changing at a rate of $5ms^{-2}$. Compute the acceleration vector of the object when it is at this point.

Solution: Use the decomposition

$$\mathbf{a} = v'\mathbf{T} + kv^2\mathbf{N}$$

where $\mathbf{T}, \mathbf{N}, k$ are the unit tangent, unit normal and curvature at the point $\langle \frac{\pi}{2}, 1\rangle$. We computed all these terms in part (a). Hence,

$$\mathbf{a} = 5\langle 1, 0\rangle + 4\langle 0, 1\rangle.$$

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