Math 32A - Fall 2021
Midterm 2, Wednesday Nov 10

Full Name: ______________________________

UID: ______________________________

Circle the name of your TA and the day of your discussion:

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Tuesday   Thursday

Instructions:

- Write your name and UID on each page.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Write your answers in the space provided.
- Calculators are not permitted.

Page	Points	Score
1	20	
2	20	
3	20	
Total:	60	

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You may use this page for scratch work. Work found on this page will not be graded.

Name: UID:

1. (20 points) Let P, Q, R be the points $(1,1,2),(1,0,1),(2,4,3)$ in $\mathbb{R}^3$.

   (a) Show that P, Q, R do not lie on one line and find the area of the triangle with these three points as vertices.

   Solution: To show that P, Q, R do not lie on one line, we need to show that the vectors **PQ** and **PR** are not parallel to each other. This is fairly straightforward to do, as you just need to check that the components are not proportional.

$$\mathbf{PQ} = \langle 0, -1, -1 \rangle, \ \mathbf{PR} = \langle 1, 3, 1 \rangle$$

   which are not scalar multiples of each other. Since these points do not lie on a line, they form a triangle. The area is half the magnitude of the cross product of the two vectors. So

$$\text{Area} = \frac{1}{2}||\mathbf{PQ} \times \mathbf{PR}|| = \frac{1}{2}||\langle 2, -1, 1 \rangle|| = \frac{1}{2}\sqrt{6}.$$

   (b) Find all three interior angles of the triangle (you may leave them in terms of inverse trigonometric functions).

   Solution: The angle between two vectors **v** and **w** is

$$\cos^{-1}\left(\frac{|\mathbf{v} \cdot \mathbf{w}|}{||\mathbf{v}|| ||\mathbf{w}||}\right).$$

   At P, the two vectors you want to look at are **PQ** and **PR** (or **QP**, **RP** which gives the same answer). So, you get an angle of

$$\cos^{-1}\left(\frac{-4}{\sqrt{2}\sqrt{11}}\right).$$

   Similarly, at Q, you want to look at the angle between the vectors $\mathbf{QP} = \langle 0, 1, 1 \rangle$ and $\mathbf{QR} = \langle 1, 4, 2 \rangle$, which is $\cos^{-1}\left(\frac{6}{\sqrt{2}\sqrt{21}}\right)$. At R, you can repeat similar computations, or you can just note that the three angles need to sum to π, so you can subtract the first two angles you obtained from π.

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2. (20 points) Let l be the line parameterized by the equation

$$\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle 0, 1, 4\rangle.$$

(a) Show that l is completely contained inside the plane with equation

$$2x + 4y - z = 7.$$

Solution: There are a couple methods you can use here. The first method is to use the parameterization to solve for x, y, z in terms of t, and check that those values satisfy the equation. So, we have $x = 1, y = 2 + t, z = 3 + 4t$ and plugging these into the equation of the plane, you get

$$2x + 4y - z = 2 + 8 + 4t - 3 - 4t = 7.$$

Another method is to note that $(1, 2, 3)$ is on the plane (by checking the equation) and then using the fact that $\langle 0, 1, 4\rangle$, the direction vector of the line, is parallel to the plane, as it is perpendicular to $\langle 2, 4, -1\rangle$, the normal vector of the plane.

(b) Let $\mathscr{P}$ now be the plane with equation $2x + 4y - z = 13$. Let P be a point on l. Explain why the perpendicular distance from P to $\mathscr{P}$ is the same for every choice of P on l, and compute this distance.

Solution: Note that the new plane is perpendicular to the original one. Since l is parallel to the original plane, it is also parallel to $\mathscr{P}$. So the perpendicular distance from any point on l to $\mathscr{P}$ is the same. So, let us now simply pick any point on the line, say $P = (1, 2, 3)$. To find the distance from P to $\mathscr{P}$, we move in the normal direction $\langle 2, 4, -1\rangle$ till we hit the new plane. So, parameterize the line through P parallel to this normal vector as

$$\mathbf{r}_2(s) = \langle 1, 2, 3\rangle + s\langle 2, 4, -1\rangle.$$

Let us now find the s value that makes the point lie on $\mathscr{P}$. Using the equation for $\mathscr{P}$, we have

$$2(1 + 2s) + 4(8 + 4s) - (3 - s) = 13$$

which gives us $21s = 6$. So $s = \frac{2}{7}$ and hence the point $Q = \left(1 + \frac{4}{7}, 2 + \frac{8}{7}, 3 - \frac{2}{7}\right)$ is on $\mathscr{P}$. The distance we are looking for is

$$||\mathbf{PQ}|| = \frac{2}{7}\sqrt{4 + 16 + 1} = \frac{2\sqrt{21}}{7} = \frac{2\sqrt{3}}{\sqrt{7}}.$$

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3. (20 points) Let C_1 be the curve in the xy-plane parameterized by the equation

$$\mathbf{r}_1(t) = \langle 1 + \cos(t), \sin(t) \rangle, \quad 0 \leq t \leq 2\pi$$

and let C_2 be the curve in the xy-plane parameterized by the equation

$$\mathbf{r}_2(u) = \langle \cos(u) - 1, \sin(u) \rangle \quad 0 \leq u \leq 2\pi.$$

(a) Show that C_1 and C_2 intersect at exactly one point.
Solution: We want to find t, u such that

$$\langle 1 + \cos t, \sin t \rangle = \langle \cos u - 1, \sin u.$$

Since $\sin t = \sin u$, $u = t$ or $u = \pi - t$. If $u = t$, then

$$1 + \cos t = \cos t - 1$$

which is impossible. Hence, $u = \pi - t$, and then

$$1 + \cos t = -\cos t - 1 \Rightarrow \cos t = -1.$$

Thus, $t = \pi$, $u = 0$ is the only point at which the two curves intersect.

(b) Let P be the point of intersection. Show that the tangent vector to C_1 at P is parallel to the tangent vector to C_2 at P.
Solution: The two tangent vectors are

$$\mathbf{r}_1(\pi) = \langle -\sin(\pi), \cos(\pi) \langle = \langle 0, -1 \rangle$$

and

$$\mathbf{r}_2(0) = \langle -\sin(0), \cos(0) \langle = \langle 0, 1 \rangle$$

which are parallel.

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