Question 1. (3 points) Let $f(x, y, z) = (x + y)^{x+z}$, where $x, y, z > 0$. Calculate the gradient of f.

$$f(xyz) = (x+y)^{x+z} = e^{\ln(x+y)^{x+z}} = e^{(x+z)\ln(x+y)}$$

$$f(xyz) = e^{(x+z)\ln(x+y)}$$

$$f_x = e^{(x+z)\ln(x+y)} \frac{d}{dx}\left[\overset{1}{(x+z)} \overset{\frac{1}{x+y}}{\ln(x+y)} \right]$$

$$= e^{(x+z)\ln(x+y)} \left[\ln(x+y) + \frac{x+z}{x+y} \right]$$

$$f_y = e^{(x+z)\ln(x+y)} \frac{d}{dy}\left[\overset{0}{(x+z)} \overset{\frac{1}{x+y}}{\ln(x+y)} \right]$$

$$= e^{(x+z)\ln(x+y)} \left[\frac{x+z}{x+y} \right]$$

$$f_z = e^{(x+z)\ln(x+y)} \frac{d}{dz}\left[\overset{1}{(x+z)} \overset{0}{\ln(x+y)} \right]$$

↳ ln(constant) = constant

$$e^{(x+z)\ln(x+y)} [\ln(x+y)]$$

$$\boxed{\nabla f = e^{(x+z)\ln(x+y)} \left\langle \ln(x+y) + \frac{x+z}{x+y}, \frac{x+z}{x+y}, \ln(x+y) \right\rangle}$$

Question 2. (5 points) Let $f(x,y)=\log(e^x+e^y)$. (Here log indicates logarithm in base e.)

i) ii)

(a) Calculate the directional derivative of $f(x,y)$ at the point (a,b) in the direction of $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. Show that the answer does not depend on a, b.

$f(xy) = \ln(e^x + e^y)$

a) directional derivative $= \nabla f \cdot \vec{u}$ $\quad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

$f_x = \left(\frac{1}{e^x+e^y}\right)(e^x)$

$f_y = \left(\frac{1}{e^x+e^y}\right)(e^y)$

$\nabla f = \left\langle \frac{e^x}{e^x+e^y}, \frac{e^y}{e^x+e^y} \right\rangle$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}{\sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}} = \frac{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}{\frac{1}{2}} = \left(\left\langle \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right\rangle\right)\frac{\sqrt{2}}{\sqrt{2}} = \left(\frac{2\sqrt{2}}{2}, \frac{2\sqrt{2}}{2}\right)$

$\frac{1}{2}+\frac{1}{2}=\frac{1}{4}$

$= \langle \sqrt{2}, \sqrt{2} \rangle$

directional derivative $= \left\langle \frac{e^x}{e^x+e^y}, \frac{e^y}{e^x+e^y} \right\rangle \cdot \langle \sqrt{2}, \sqrt{2} \rangle$

$$\boxed{\text{directional derivative} = \frac{\sqrt{2}e^x + \sqrt{2}e^y}{e^x+e^y}}$$

not dependent @ point (a,b) $\longrightarrow$ $\frac{\sqrt{2}\,\cancel{(e^x+e^y)}}{\cancel{(e^x+e^y)}}$ $\boxed{= \sqrt{2}}$

i) ii)

(b) Find the equation of the line through the points $A=(0,0,\log(2))$ and $B=(1,1,\log(2e))$ and show that it is entirely contained in the graph of f.

b) $\vec{v} = \vec{AB}$

$\vec{v} = \langle a,b,c \rangle$

$z = z_0 + ct$

gen eqn line: $\langle \underbrace{x_0, y_0, z_0}_{\text{point}} \rangle + t\langle \underbrace{a,b,c}_{\text{line}} \rangle$

$\vec{V} = \langle 1-0, 1-0, \ln(2e) - \ln(2) \rangle$

$= \langle 1, 1, \ln 2 + 1 - \ln 2 \rangle$

$= \langle 1, 1, 1 \rangle$

$$\boxed{\begin{aligned}\text{equation of line: } &\langle 0,0,\ln 2 \rangle + t\langle 1,1,1 \rangle \\ &\langle 1,1,\log(2e) \rangle + t\langle 1,1,1 \rangle\end{aligned}}$$

<u>entirely contained in line</u>

@ point A:

$f(x,y) = \ln(2)$

@ point B:

$f(x,y) = \ln(2e)$

points also on plane = pt from line, so must be contained on f

$f(x,y) = z$ also equals those values $\longrightarrow$ so line must be on graph

Question 3. (5 points) Let $f(x,y) = xye^{-2x-y}$. Find all critical points of f, and determine if they are local maxima, local minima or saddle points.

det

$f(xy) = xye^{-2x-y}$

critical points — gradient = (One, One) (0,0) (One, 0) (0, One)

$f(xy) = yx\Big|e^{-2x-y}$ (y, $-2e^{-2x-y}$)

$f(xy) = xy\Big|e^{-2x-y}$ (x, $-e^{-2x-y}$)

$f_x = ye^{-2x-y} - 2yxe^{-2x-y}$
$= e^{-2x-y}[y - 2yx] = 0$
$y - 2yx = 0$
$y(1-2x) = 0$
$y = 0$ or $1 - 2x = 0$
$x = 1/2$

$f_y = xe^{-2x-y} - xye^{-2x-y}$
$= e^{-2x-y}[x - xy] = 0$
$x - xy = 0$
$x(1-y) = 0$
$x = 0$ or $y = 1$

$$\nabla f \begin{cases} f_x = 0, & y = 0 \text{ or } x = 1/2 \\ f_y = 0, & x = 0 \text{ or } y = 1 \end{cases} \rightarrow \boxed{\begin{matrix}(0,0)\\(1/2, 1)\end{matrix}}$$

critical pts

local max / min / saddle pt

2nd Derivative Test

$P = (a,b)$ f_{xx}, f_{yy}, f_{xy} exist and cont. near P

1. $D > 0$ $f_{xx}(ab) < 0$ local max
2. $D > 0$ $f_{xx}(ab) > 0$ local min
3. $D < 0$ neither (saddle point)
4. $D = 0$ test fails

discriminant

$D = f_{xx}(ab) f_{yy}(ab) - [f_{xy}(ab)]^2$

(0,0)

$D = (0)(0) - [1]^2 = -1$

$\boxed{D < 0 \text{ saddle point @ } (0,0)}$

(1/2, 1)

$D = (-\frac{2}{e^2})(-\frac{1}{2e^2}) = \frac{1}{e^4} \approx 0.018$

$D > 0$

$f_{xx} = -\frac{2}{e^2} \approx -0.27$

$f_{xx} < 0$

$\boxed{\text{Because } D > 0 \text{ and } f_{xx} < 0 \text{ local max @ } (1/2, 1)}$

$f_x = e^{-2x-y}\Big|[y - 2yx]$ ($-2e^{-2x-y}$, $-2y$)

$f_{xx} = -2e^{-2x-y}(y - 2yx) - 2ye^{-2x-y}$
$= e^{-2x-y}[-2(y - 2yx) - 2y]$
$= e^{-2x-y}[-2y + 4yx - 2y]$
$= e^{-2x-y}[4yx - 4y]$

$f_{xx}(0,0) = e^0[0] = 0$

$f_{xx}(1/2, 1) = e^{-1-1}[(4)(1)(1/2) - (4)(1)]$ ($2 - 4$)
$= \frac{-2}{e^2}$

$f_y = e^{-2x-y}\Big|[x - xy]$ ($-e^{-2x-y}$, $-x$)

$f_{yy} = -e^{-2x-y}[x - xy] - xe^{-2x-y}$
$= e^{-2x-y}[-x + xy - x]$
$= e^{-2x-y}[xy - 2x]$

$f_{yy}(0,0) = e^0[0] = 0$

$f_{yy}(1/2, 1) = e^{-1-1}[(1/2)(1) - (2)(1/2)]$ ($1/2 - 2/2$)
$= \frac{-1}{2e^2}$

$f_{xy} = \frac{d}{dy}[e^{-2x-y}\Big|(y - 2yx)]$ ($-e^{-2x-y}$, $1 - 2x$)
$= -e^{-2x-y}(y - 2yx) + (1 - 2x)e^{-2x-y}$
$= e^{-2x-y}[-y + 2yx + 1 - 2x]$

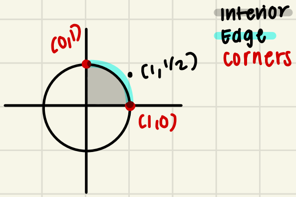
$f_{xy}(0,0) = e^0[1] = 1$

$f_{xy}^2 = 1$

$f_{xy}(\frac{1}{2}, 1) = e^{-2}[-1 + (2)(1)(1/2) + 1 - 2(1/2)]$ ($-1 + 1 + 1 - 1$)
$= 0$

$f_{xy}^2 = 0$

Question 4. (5 points) Find the global maxima and minima of $f(x,y) = x^2 + y^2 - 2x - y$ on the domain $\boxed{\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.}$



$\{(x,y) \in \mathbb{R}^2 : \underbrace{x^2+y^2 \leq 1}_{\text{unit circle}}, x \geq 0, y \geq 0\}$

Interior $\nabla f = (0,0)$

$\nabla f = \langle 2x-2, 2y-1 \rangle$

$2x - 2 = 0$
$x = 1$

$2y - 1 = 0$
$y = 1/2$

$(1, 1/2) \longrightarrow$ not in domain

$1 + \frac{1}{4}$ is not ≤ 1

Edges

$\ell = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1, x \geq 0, y \geq 0\} = \{(x,y) : y = \sqrt{1-x^2}, 0 \leq x \leq 1\}$

$g(x) = f(x, \sqrt{1-x^2}) = \cancel{x^2} + 1 - \cancel{x^2} - 2x - \sqrt{1-x^2}$

$= 1 - 2x - (1-x^2)^{1/2}$

$g'(x) = 2 - \cancel{\frac{1}{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$

$= 2 - \dfrac{x}{(1-x^2)^{1/2}}$

CP $g'(x)$: $\quad 2 - \dfrac{x}{(1-x^2)^{1/2}} = 0$

$\dfrac{x}{(1-x^2)^{1/2}} = 2$

$x = 2(1-x^2)^{1/2}$

$\dfrac{x}{(1-x^2)^{1/2}} = 2 \qquad \dfrac{\sqrt{\frac{4}{5}}}{\underbrace{(1-\frac{4}{5})^{1/2}}_{=1/5}} = \dfrac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} = \frac{2}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{\sqrt{5}}}{1} = 2$

calculator $\quad x = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

$y = \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}} \qquad (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) \longrightarrow$ in domain

$\frac{4}{5} + \frac{1}{5} = 1$ which is ≤ 1
and $x \geq 0, y \geq 0$

$f(x,y) = x^2 + y^2 - 2x - y$

$f(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \frac{4}{5} + \frac{1}{5} - \frac{4}{\sqrt{5}} - \frac{1}{\sqrt{5}}$

$= 1 - \frac{5}{\sqrt{5}} = 1 - \sqrt{5} \approx -1.24$

corners $\quad f(x,y) = x^2 + y^2 - 2x - y$

$f(1,0) = 1 - 2 = -1$

$f(0,1) = 1 - 1 = 0$

Possible Global Min / Max

$\boxed{f(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = 1 - \sqrt{5} \ (\approx -1.24) \longrightarrow \text{global min}}$

$f(1,0) = -1$

$\boxed{f(0,1) = 0 \longrightarrow \text{global max}}$

Question 5. (3 points per limit) Compute the following limits:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2 + x^2y^4}{x^6 + y^6} = \frac{x^2y^2\,[\,x^2 + y^2\,]}{x^6 + y^6}$$

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(xy)}{x^2 + y^2}$$

Hint: for the second limit, use the inequality $|\sin(t)| \leq |t|$, valid for all real numbers t. (You do not need to prove this inequality.)

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2 + x^2y^4}{x^6+y^6} = \frac{x^2y^2\,[x^2+y^2]}{x^6+y^6}$ $\quad y = mx$

$\displaystyle\lim_{x\to 0} \frac{x^2m^2x^2\,[x^2+m^2x^2]}{x^6+m^6x^6} = \frac{x^{\cancel{4}\,2}m^2\,[x^2+m^2x^2]}{\cancel{x^6}\,[1+m^6]} = \frac{m^2x^2+m^4x^2}{[1+m^6]}$

$\boxed{\text{m-dependent so lim dne}}$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2\,[x^2+y^2]}{x^6+y^6}$ $\quad y=x$

$\displaystyle\lim_{x\to 0} \frac{x^4\,[2x^2]}{2x^6} = \frac{2x^6}{2x^6} = 1$

$\boxed{\text{Because limit along } y=x \neq \text{limit along } y=x^2 \ \lim_{(x,y)\to(0,0)} \text{DNE}}$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2\,[x^2+y^2]}{x^6+y^6}$ $\quad y=x^2$

$\displaystyle\lim_{x\to 0} \frac{x^2x^4\,\overset{x^2[1+x^2]}{[x^2+x^4]}}{x^6+x^{12}} = \frac{x^8\,(1+x^2)}{x^6\,(1+x^6)} = \frac{x^2(1+x^2)}{(1+x^6)} = \frac{0}{1} = 0$

Question 5. (3 points per limit) Compute the following limits:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2 + x^2y^4}{x^6 + y^6}$$

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(xy)}{x^2 + y^2}$$

Hint: for the second limit, use the inequality $|\sin(t)| \le |t|$, valid for all real numbers t. (You do not need to prove this inequality.)

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(xy)}{x^2+y^2} \qquad x = r\cos\theta \quad y = r\cos\theta$$

$$\lim_{r\to 0} \frac{r\cos\theta \sin[r\cos\theta r\sin\theta]}{\underset{\underset{=1}{r^2[\cos^2\theta + \sin^2\theta]}}{r^2\cos^2\theta + r^2\sin^2\theta}} = \frac{\cancel{r}\cos\theta \sin[r^2\cos\theta\sin\theta]}{r^{\cancel{2}}} = \frac{\cos\theta\sin[r^2\cos\theta\sin\theta]}{r} =$$

$$\frac{\cos(0)\sin(0)}{0} = \frac{0}{0} \quad \text{use LH}$$

$$-\sin\theta \quad \cos\left[\frac{r^2}{2}\sin 2\theta\right]\left[\frac{\cancel{2}r}{\cancel{2}}\right]\left[2\cos 2\theta\right]$$

$$\lim_{r\to 0} \frac{\cos\theta \sin[r^2 \cos\theta\sin\theta]}{r} = \frac{\cos\theta \,|\, \sin\left[\frac{r^2}{2}\sin 2\theta\right]}{r}$$

LH

$$\lim_{r\to 0} \frac{-\sin\theta\sin\left[\frac{r^2}{2}\sin 2\theta\right] + \cos\left[\frac{r^2}{2}\sin 2\theta\right][2\cos 2\theta][r][\cos\theta]}{1}$$

$$= \lim_{r\to 0} \frac{-\sin\theta\sin(0) + \cos(0)\, 2\cos 0\, (0)\cos\theta}{1} = \frac{0+0}{1} = \boxed{0}$$

Question 6. (3 points per subquestion) (a) Find a differentiable function $f(x, y)$ such that

$$\nabla f = \langle -2x + 3x^2y + y^2, x^3 + 2xy\rangle$$

or show that it does not exist.

(b) Find a differentiable function $f(x, y)$ such that $\nabla f = \langle 3x^2y + y^2, x^3\rangle$ or show that it does not exist.

a) $\nabla f = \langle -2x + 3x^2y + y^2, x^3 + 2xy \rangle$

$f_{xy} = \frac{d}{dy}(f_x) = \frac{d}{dx}(f_y)$

$f_x = -2x + 3x^2y + y^2$

$f_y = x^3 + 2xy$

wrt x: $\int -2x + 3x^2y + y^2 \, dx$

$= -x^2 + x^3y + y^2x + c$

wrt y: $\int x^3 + 2xy \, dy$

$= yx^3 + xy^2 + c$ → could be x^3 since $\frac{d}{dy}x^3$, x^3 is constant

$\boxed{\text{possible diff eq: } x^3y + y^2x - x^2 + c}$

b) $\nabla f = \langle 3x^2y + y^2, x^3 \rangle$

$f_x = 3x^2y + y^2$

$f_y = x^3$

wrt x: $\int 3x^2y + y^2 \, dx$

$= yx^3 + \boxed{y^2x} + c$ → not in integral w/ respect to y of f_y

wrt y: $\int x^3 \, dy$

$= x^3y + c$ → cannot be y^2x since $\frac{d}{dy} = 2yx$ and integral w/ respect to y doesn't have $2yx$

$\boxed{\text{DNE}}$