Quiz 2 Solutions

Math 32A
UCLA

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Question 1

Problem

A particle moves in the xy-plane according to the parametrization $\vec{r}(t) = \langle t^2 - 1, t^3 - t \rangle$, where t is a real number. Find the two times $t_1 < t_2$ at which the particle passes through the origin $(0, 0)$. Compute velocity vector, speed and acceleration vector of the particle at the times t_1 and t_2.

Solution

To find t_1 and t_2 we set $\vec{r}(t) = \langle 0, 0 \rangle$. This gives $t^2 - 1 = 0$ and $t^3 - t = 0$. The solutions to the first equation are $t = -1, 1$, and the solutions to the second are $t = -1, 0, 1$. Since both equations need to be true we have $t_1 = -1$ and $t_2 = 1$.

We can compute $\vec{v}(t) = \vec{r}'(t) = \langle 2t, 3t^2 - 1 \rangle$. Taking the magnitude gives

$$v(t) = ||\langle 2t, 3t^2 - 1 \rangle|| = \sqrt{(2t)^2 + (3t^2 - 1)^2} = \sqrt{9t^4 - 2t^2 + 1}.$$

And we can also compute $\vec{a}(t) = \vec{v}'(t) = \langle 2, 6t \rangle$.

Therefore $\vec{v}(t_1) = \langle -2, 2 \rangle$, $\vec{v}(t_2) = \langle 2, 2 \rangle$, $v(t_1) = 2\sqrt{2}$, $v(t_2) = 2\sqrt{2}$, $\vec{a}(t_1) = \langle 2, -6 \rangle$, and $\vec{a}(t_2) = \langle 2, 6 \rangle$.

Question 2

The hyperbola of equation $y^2 - x^2 = 1$ consists of two branches. Find the point (or points) of maximum curvature of the upper branch of $y^2 - x^2 = 1$ (that is, the branch contained in the upper half-plane $y \geq 0$) and compute the curvature at that point (or those points).

Solution

First we show that $\overrightarrow{r}(t) = \langle \frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2} \rangle$ parametrizes the upper branch. For this we verfy

$$\left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \left(\frac{1}{4}e^{2t} + \frac{1}{2} + \frac{1}{4}e^{-2t}\right) - \left(\frac{1}{4}e^{2t} - \frac{1}{2} + \frac{1}{4}e^{-2t}\right) = 1.$$

This shows that $\overrightarrow{r}(t)$ lies on the hyperbola. And the fact that $\frac{e^t + e^{-t}}{2} > 0$ shows that $\overrightarrow{r}(t)$ lies on the upper branch. Lastly, because $\frac{e^2 - e^{-t}}{2}$ is increasing from $-\infty$ to ∞, we parametrize the whole upper branch.

Now we compute $\overrightarrow{v}(t) = \langle \frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \rangle$. And so

$$||\overrightarrow{v}(t)||^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 + \left(\frac{e^t - e^{-t}}{2}\right)^2 = \left(\frac{1}{4}e^{2t} + \frac{1}{2} + \frac{1}{4}e^{-2t}\right) + \left(\frac{1}{4}e^{2t} - \frac{1}{2} + \frac{1}{4}e^{-2t}\right) = \frac{e^{2t} + e^{-2t}}{2}.$$

Also we have $\overrightarrow{a}(t) = \langle \frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2} \rangle$. Therefore

$$||\overrightarrow{v}(t) \times \overrightarrow{a}(t)||^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = 1.$$

This implies

$$\kappa(t) = \frac{||\overrightarrow{v}(t) \times \overrightarrow{a}(t)||}{||\overrightarrow{v}||^3} = \left(\frac{e^{2t} + e^{-2t}}{2}\right)^{-\frac{3}{2}}.$$

And thus $\kappa(t)$ is maximized when $\frac{e^{2t} + e^{-2t}}{2}$ is minimized. We see $\frac{d}{dt}(\frac{e^{2t} + e^{-2t}}{2}) = e^{2t} - e^{-2t}$. Setting this to zero gives $e^{2t} = e^{-2t}$, so $e^{4t} = 1$. Taking a natural logarithm gives $4t = \ln(1) = 0$, so $t = 0$.

Plugging this in gives that the maximum value of the curvature is 1 at the point $(0, 1)$.