MATH 32A LEC 2 - MIDTERM 2 SOLUTIONS

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Question 1. (4 points) Show that the curves parametrized by $\vec{r}_1(s) = \langle e^s, e^{2s}, 1 - e^{-s} \rangle$ and $\vec{r}_2(t) = \langle 1 - t, \cos(t), \sin(t) \rangle$ intersect at the point $P = (1, 1, 0)$, and find the angle between their tangent vectors at P.

Solution. We have $\vec{r}_1(s) = \langle 1, 1, 0 \rangle$ if and only if $s = 0$, and similarly $\vec{r}_2(t) = \langle 1, 1, 0 \rangle$ if and only $t = 0$. Thus the curves intersect at P, and each curve passes through P exactly one time (when $s = 0$ and $t = 0$).

We have

$$\vec{r}_1'(0) = \left\langle e^s, 2e^{2s}, e^{-s} \right\rangle |_{s=0} = \langle 1, 2, 1 \rangle$$

and

$$\vec{r}_2'(0) = \langle -1, -\sin(t), \cos(t) \rangle |_{t=0} = \langle -1, 0, 1 \rangle .$$

Let θ be the angle between the tangent vectors. Then

$$\cos(\theta) = \frac{\langle 1, 2, 1 \rangle \cdot \langle -1, 0, 1 \rangle}{| \langle 1, 2, 1 \rangle | \cdot | \langle -1, 0, 1 \rangle |} = 0.$$

Thus the answer is $\theta = \pi/2$. □

Question 2. (3 points per limit.) Compute the following limits or show that they do not exist.

(1) $$\lim_{(x,y)\to(0,0)} \frac{x^4 y^4}{(x^2 + y^4)^3}.$$

(2) $$\lim_{(x,y)\to(0,0)} \frac{\log(1 + xy)}{xy} \cdot \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2}.$$

(As always, log indicates logarithm in base e.)

Solution. The first limit does not exist. Indeed, approaching the origin via the curve of equation $x = my^2$, where m is any real number, the limit reduces to

$$\lim_{y\to 0} \frac{m^4 y^8 \cdot y^4}{(m^2 y^4 + y^4)^3} = \lim_{y\to 0} \frac{m^4 y^{12}}{(m^2 + 1)^3 y^{12}} = \lim_{y\to 0} \frac{m^4}{(m^2 + 1)^3} = \frac{m^4}{(m^2 + 1)^3}.$$

This value depends on m. For example, it is 0 when $m = 0$, and it is $1/8$ when $m = 1$. Therefore, the original limit cannot exist.

The second limit is equal to $-1/2$. Indeed, write $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (polar coordinates). Then

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2} = \lim_{r\to 0} \frac{1 - \cos(r^2)}{r^4}.$$

Call $u = r^2$, then

$$\lim_{r\to 0} \frac{1-\cos(r^2)}{r^4} = \lim_{u\to 0} \frac{1-\cos(u)}{u^2} = \lim_{u\to 0} \frac{\sin(u)}{2u} = \lim_{u\to 0} \frac{\cos(u)}{2} = \frac{1}{2},$$

where we have used L'Hôpital's Rule twice.

The limit of the other fraction can also be computed using polar coordinates, but We proceed in a slightly easier way. Call $z = xy$: if $(x, y) \to 0$, then $z \to 0$. Then

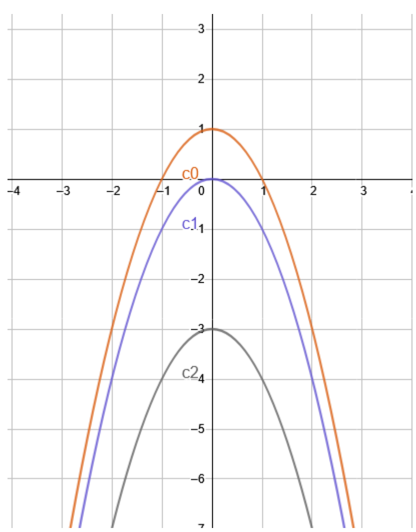
$$\lim_{(x,y)\to(0,0)} \frac{\log(1+xy)}{xy} = \lim_{z\to 0} \frac{\log(1+z)}{z} = \lim_{z\to 0} \frac{1/(z+1)}{1} = 1,$$

where we have used L'Hôpital's Rule once. We conclude that the original limit exists and equals $1 \cdot \frac{1}{2} = \frac{1}{2}$. □

Question 3. (a) (3 points) Let $f(x, y) = \sqrt{1 - x^2 - y}$. Describe all level curves for f. Draw a contour map containing at least 3 distinct level curves.

(b) (3 points) Consider the surface of equation $zx - zy - x - y = 0$. Describe all horizontal traces $z = c$, and draw the level curves for $c = -2, -1, 0, 1, 2$.

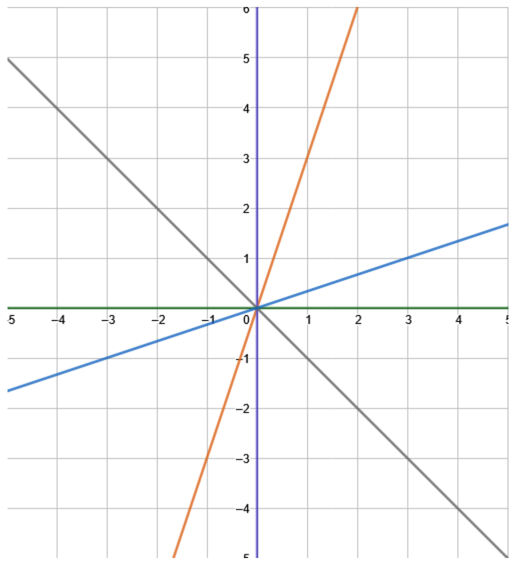
Solution. (a) The graph of f is given by the equation $z = \sqrt{1 - x^2 - y}$. Since a square root is never negative, the level curves $f(x, y) = c$ are empty when $c < 0$. When $c \geq 0$, the levels curve are given by $c = \sqrt{1 - x^2 - y}$, or equivalently $c^2 = 1 - x^2 - y$. These are parabolas with vertical axis, since they can be rewritten as $y = -x^2 - c^2 + 1$. Here is a picture the level curves corresponding to $c = 0, 1, 2$:



(b) The level curves are of the form $cx - cy - x - y = 0$, which can be rewritten as $(c+1)y = (c-1)x$. Therefore the level curves are lines through the origin: a vertical line if $c = -1$, and a line of slope $\frac{c-1}{c+1}$ otherwise. Here is a picture: □

Question 4. (4 points) Let $f(r, \theta) = r^n \sin(n\theta)$, where $n \geq 1$ is a natural number. Show that

$$f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0.$$



Solution. We calculate

$$f_r = nr^{n-1}\sin(n\theta), \qquad f_{rr} = n(n-1)r^{n-2}\sin(n\theta)$$

and

$$f_\theta = nr^n\cos(n\theta), \qquad f_{\theta\theta} = -n^2r^n\sin(n\theta).$$

Therefore

$$\begin{aligned} f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta} &= n(n-1)r^{n-2}\sin(\theta) + nr^{n-2}\sin(\theta) - n^2r^{n-2}\sin(\theta) \\ &= (n(n-1) + n - n^2)r^{n-2}\sin(\theta) \\ &= 0 \cdot r^{n-2}\sin(\theta) = 0. \end{aligned}$$

□