MATH 32A - MIDTERM 1 - SOLUTIONS

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Question 1 (3 points)**.** Let $\vec{v}_1 = \langle -3, -4, 2\rangle$ and $\vec{v}_2 = \langle 1, 7, 1\rangle$. Find the only vector $\vec{v}$ perpendicular to $\vec{k}$ such that $\vec{v}_1 + \vec{v}_2 + \vec{v}$ is parallel to the z-axis.

Solution. Let $\vec{v} = \langle a, b, c\rangle$. The fact that $\vec{v}$ is perpendicular to $\vec{k} = \langle 0, 0, 1\rangle$ translates to

$$0 = \vec{v} \cdot \vec{i} = \langle a, b, c\rangle \cdot \langle 0, 0, 1\rangle = c,$$

so $\vec{v} = \langle a, b, 0\rangle$. We have

$$\vec{v}_1 + \vec{v}_2 + \vec{v} = \langle -3, -4, 2\rangle + \langle 1, 7, 1\rangle + \langle a, b, 0\rangle = \langle -2 + a, 3 + b, 3\rangle .$$

The vector $\langle -2 + a, 3 + b, 3\rangle$ is parallel to $\vec{k}$ if and only if

$$\langle -2 + a, 3 + b, 3\rangle = \langle 0, 0, \lambda\rangle$$

for some scalar λ. The only solution to the above equality is $\lambda = 3$, $a = 2$ and $b = -3$. Therefore $\vec{v} = \langle 2, -3, 0\rangle$. $\square$

Question 2 (3 points)**.** Starting from the time $t = 0$, a particle moves in space following the curve given by $c(t) = (\sqrt{t} + 1, t, -\sqrt{t})$. Let $P_0 = c(0)$ denote the starting point of the particle.

(1 point) At what time is the distance between the particle and P_0 equal to $\sqrt{3}$?

(1 point per projection) Find the equations that describe the xy-projection and the xz-projection of $c(t)$ and sketch their graphs. (Label the axes and don't forget that $t \geq 0$.)

Solution. (i) By plugging $t = 0$ into the definition of $c(t)$, we see that $P_0 = c(0) = (1, 0, 0)$. At any time t, the square of the distance between the point $c(t)$ and $(1, 0, 0)$ is given by

$$(\sqrt{t} + 1 - 1)^2 + (t - 0)^2 + (-\sqrt{t} - 0)^2 = t + t^2 + t = t^2 + 2t.$$

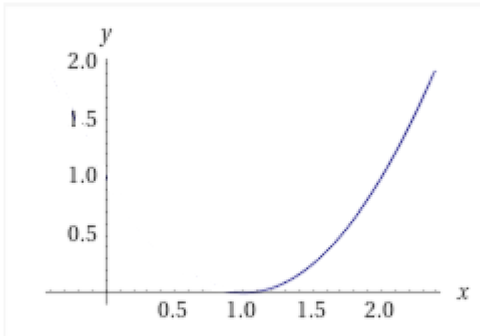
(Here we have used the fact that $(\sqrt{t})^2 = t$ when $t \geq 0$.) We want this to be equal to $(\sqrt{3})^2 = 3$, that is, we want to find the solutions of $t^2 + 2t - 3 = 0$. These are $t = 1$ and $t = -3$, but only $t = 1$ is valid because we are assuming $t \geq 0$. Therefore $t = 1$.

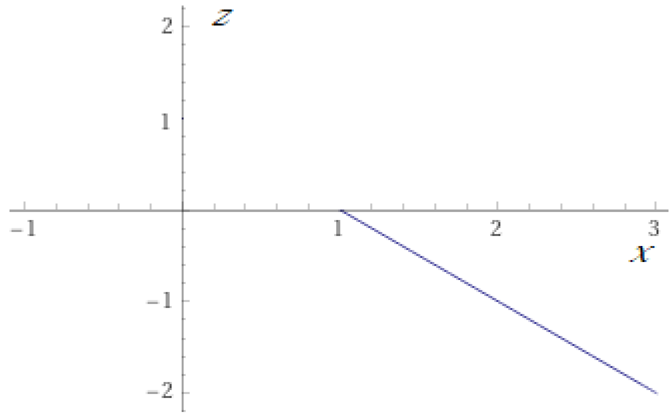
(ii) The xy-projection of $c(t)$ is given by $x(t) = \sqrt{t} + 1$ and $y(t) = t$, which implies $x = \sqrt{t} + 1 = \sqrt{y} + 1$ and $y \geq 0$. Thus the xy-projection of $c(t)$ is described by

$$x = \sqrt{y} + 1, \qquad y \geq 0.$$

The xz-projection of $c(t)$ is given by $x(t) = \sqrt{t} + 1$ and $z(t) = -\sqrt{t}$, which implies $x = \sqrt{t} + 1 = -z + 1$, and so the xy-projection of $c(t)$ is described by

$$x = -z + 1, \qquad z \geq 0. \qquad \square$$





Question 3 (4 points)**.** Let $A = (1, 0, 2)$, $B = (2, 3, 5)$, $C = (-1, -1, -1)$ and $D = (0, 0, 3)$. Let ℓ be the line obtained by intersecting the plane passing through A, B, C and the plane passing through B, C, D.

(2 points) Find a parametrization of the line ℓ.

(2 points) Find the equation of the plane perpendicular to ℓ and passing through $P = (2, -2, 0)$.

Solution. (i) The line ℓ passes through B and C. We have

$$\overrightarrow{BC} = \langle -1-2, -1-3, -1-5 \rangle = \langle -3, -4, -6 \rangle .$$

Therefore a possible parametrization for ℓ is given by

$$\vec{r}(t) = \langle 2, 3, 5 \rangle + t \langle -3, -4, -6 \rangle .$$

(Of course, there are many other correct answers, for example $\vec{r}(t) = \langle 2, 3, 5 \rangle + t \langle 3, 4, 6 \rangle$.)

(ii) We must write the equation of the plane through $(2, -2, 0)$ and with normal direction $\langle -3, -4, -6 \rangle$. From the general formula, we obtain:

$$-3(x-2) - 4(y+2) - 6(z-0) = 0,$$

which we can also rewrite as

$$-3x - 4y - 6z = 2$$

or if you prefer

$$3x + 4y + 6z = -2. \qquad \square$$

Question 4 (5 points)**.** Two particles move in the plane starting from the time $t = 0$ and according to the vector parametrizations $\vec{r}_1(t) = \langle \cos(2\pi t), -\sin(2\pi t) \rangle$ and $\vec{r}_2(t) = \left\langle t, \frac{t^2-1}{2} \right\rangle$, where $t \geq 0$.

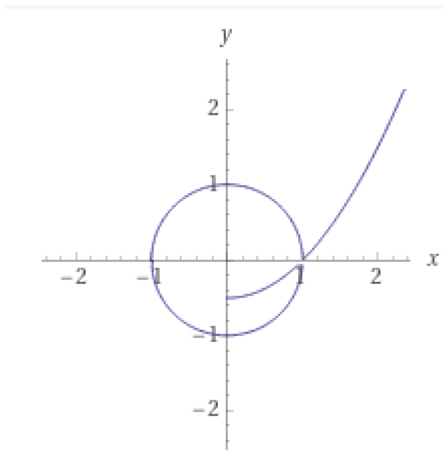
(2 points) Describe the curves parametrized by $\vec{r}_1(t)$ and $\vec{r}_2(t)$ using only the coordinates x and y. (In class we called this procedure *elimination of the parameter* t.)

(1 point) Draw the two curves in the same cartesian plane. (Careful, we are supposing $t \geq 0$.)

(2 points) Do the particles collide? If so, find the time t of collision.

Solution. (i) The parametrization $\vec{r}_1(t)$ describes the unit circle $x^2 + y^2 = 1$. The parametrization $\vec{r}_1(t)$ describes the right half of a parabola. More precisely, we have $x = t \geq 0$ and $y = \frac{t^2-1}{2}$. By substitution, we obtain $y = \frac{x^2-1}{2}$ and $x \geq 0$. Therefore $\vec{r}_2(t)$ describes the part of the parabola $y = \frac{x^2-1}{2}$ where $x \geq 0$.

(ii) Using the information obtained in part (i), we can draw the following picture:



(iii) We begin by finding the point of intersection between the paths of the two particles. One immediately sees from the picture that the only point of intersection is $(1, 0)$, so no calculations are required. However, one can also proceed algebraically as follows. We must solve the system of equations

$$x^2 + y^2 = 1, \quad y = \frac{x^2 - 1}{2}.$$

The second equation can be rewritten as $x^2 = 2y + 1$. Substituting this into the first equation, we obtain $2y + 1 + y^2 - 1 = 0$, that is, $y^2 + 2y = 0$. This implies that either $y = 0$, which yields $x = \pm 1$, or $y = -2$, which yields $x^2 = -3$, impossible. The condition $x \geq 0$ now implies that the only solution is $(1, 0)$.

So far we only know that the two paths cross at $(1, 0)$. If the particles passed through $(1, 0)$ at different times, there would be no collision. The second particle passes through $(1, 0)$ when $t = x = 1$. When $t = 1$, the position of the first particle is given by $(\cos(2\pi), -\sin(2\pi)) = (1, 0)$. Therefore the two particles collide at $t = 1$. □

Question 5 (5 points)**.** The curve parametrized by $c(t) = (t, t^2, t^3)$ intersects the plane of equation $12x - 7y + z = 0$ in three distinct points A, B and C.

(2 points) Find the area of the triangle ABC.

(3 points) Let $D = c(2) = (2, 4, 8)$. Find the volume of the parallelepiped generated by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

Solution. The curve $c(t)$ intersects the plane of equation $12x - 7y + z = 0$ exactly when

$$12t - 7t^2 + t^3 = 0.$$

This is equivalent to $t(t^2 - 7t + 12) = t(t - 3)(t - 4) = 0$, and so has three solutions $t = 0, 3, 4$. Therefore $A = (0, 0, 0)$, $B = (3, 9, 27)$ and $C = (4, 16, 64)$. (Any other ordering is also correct.) Using the formula for the area of a triangle, we know that the area of the triangle ABC is equal to

$$\frac{1}{2}||\,\langle 3,9,27\rangle \times \langle 4,16,64\rangle\,|| = 6||\,\langle 1,3,9\rangle \times \langle 1,4,16\rangle\,|| = 6\sqrt{194}.$$

The volume of the parallelepiped generated by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ is given by the absolute value of

$$\begin{vmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \end{vmatrix} = 2\cdot 3\cdot 4\cdot \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} = 24\cdot((48-36)-2(16-9)+4(4-3)) = 24\cdot(12-14+4) = 48.$$

Therefore the volume is equal to 48. (One could also use the formula which expresses the volume as the absolute value of $\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})$, and then perform similar computations.) $\square$