Name: Answer Key

SID:

1. Let $f(x) = e^x + x$.

   (a) (5 pts) Show that $f(x)$ is invertible.

   $f'(x) = e^x + 1$ so $f'(x) > 0$ for all x since $e^x > 0$ for all x and so $e^x + 1 > 1$ for all x.

   so $f(x)$ is strictly increasing for all x, so $f(x)$ is invertible

   (b) (5 pts) Find $g'(1)$, where g is the inverse of f.

   $f(0) = e^0 + 0 = 1$, so $g(1) = 0$

   $$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{e^0 + 1} \boxed{= \frac{1}{2}}$$

2. (10 pts) Find

$$\frac{d}{dx}\left(x^{\cos(x)}\right)$$

$$\frac{d}{dx}\left(x^{\cos(x)}\right) = \frac{d}{dx}\left(e^{\ln\left(x^{\cos(x)}\right)}\right) = \frac{d}{dx}\left(e^{\cos(x)\ln(x)}\right)$$

$$= \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) e^{\cos(x)\ln(x)}$$

$$\boxed{= \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) x^{\cos(x)}}$$

or using logarithmic differentiation,

$$\frac{d}{dx}\ln\left(x^{\cos(x)}\right) = \frac{d}{dx}\left(\cos(x)\ln(x)\right) = \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$$

so $\frac{d}{dx}\left(x^{\cos(x)}\right) = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$

Name:

SID:

3. Evaluate the following limits:

(a) (10 pts)

$$\lim_{x\to 0} \frac{\sin(x)}{e^{2x}-1}$$

$\sin(0) = 0$ and $e^0 - 1 = 0$, so we can apply L'Hopital. $\frac{0}{0}$

$$\lim_{x\to 0} \frac{\sin(x)}{e^{2x}-1} = \lim_{x\to 0} \frac{\cos(x)}{2e^{2x}} = \frac{(1)^2}{2e^0} \boxed{= \frac{1}{2}}$$

$\sec(0) = 1$

(b) (10 pts)

$$\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x$$

$$= \lim_{x\to\infty} e^{\ln\left(\left(1-\frac{1}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1-\frac{1}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)}$$

now $\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{x^{-1}}$ $\frac{0}{0}$, so can apply L'Hopital

$$= \lim_{x\to\infty} \frac{x^{-2}\frac{1}{1-\frac{1}{x}}}{-x^{-2}}$$

$$= \lim_{x\to\infty} -\frac{1}{1-\frac{1}{x}} = -1$$

so $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)} \boxed{= e^{-1}}$

Name:

SID:

4. Evaluate the following integrals:

(a) (5 pts)

$$\int xe^{x^2}\,\mathrm{d}x$$

u substitution,

$u = x^2$

$du = 2x\,dx$

$$= \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + C$$

$$= \frac{1}{2} e^{x^2} + C$$

(b) (10 pts)

$$\int_0^1 x^2 e^x\,\mathrm{d}x$$

integration by parts

$u = x^2 \quad du = 2x\,dx$

$dv = e^{2x}\,dx \quad v = \frac{1}{2} e^{2x}$

$$= x^2 e^x - \int e^x \cdot 2x\,dx$$

integration by parts again,

$u = x \quad du = dx$

$dv = 2e^x\,dx \quad v = 2e^x$

$$= x^2 e^x - \left(x \cdot 2e^x - \int 2e^x\,dx \right)$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

So $\int_0^1 x^2 e^x = \left(x^2 e^x - 2xe^x + 2e^x \right)\Big|_0^1 = \left(e^1 - 2e^1 + 2e^1\right) - (2 \cdot 1)$

$$\boxed{= e - 2}$$

(c) (10 pts)

$$\int \tan^{-1}(x)\,\mathrm{d}x$$

integration by parts,

$u = \tan^{-1}(x) \quad du = \frac{1}{1+x^2}$

$dv = dx \quad v = x$

$$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx = x\tan^{-1}(x) - \int \frac{1}{2}\frac{1}{u}\,du$$

u substitute $u = 1 + x^2$

$du = 2x\,dx$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2| + C$$

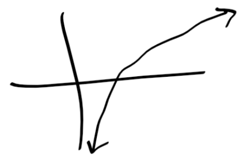
Name:

SID:

5. True/False Circle the correct answer (2 pts each). You do not need to justify your answers.

(a) **True** / **False**: $(e^a)^b = e^{(a^b)}$ for all $a > 0$ and $b > 0$

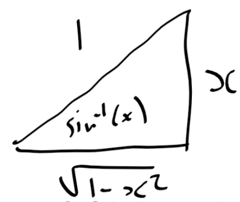
False, $(e^a)^b = e^{ab} \neq e^{(ab)}$ in general (eg. $a=1, b=2$)



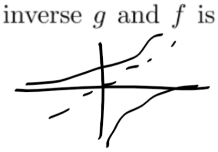
(b) **True** / **False**: $\lim_{x\to\infty} \ln(x) = 0$.

$= \infty$

(c) **True** / **False**: $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$.



(d) **True** / **False**: If f is one-to-one with inverse g and f is increasing on its domain, then g is increasing on its domain.



if $a \leq b$ $\underbrace{f(a)}_{c} \leq \underbrace{f(b)}_{d}$

so if $c \leq d$, $\underbrace{g(c)}_{a} \leq \underbrace{g(d)}_{b}$

(e) **True** / **False**: If f is differentiable, then $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$

integration by parts applied to

$u = f(x)$ $\quad dv = dx$

$du = f'(x)$ $\quad v = x$