Midterm 2
Math 31B

Spring, 2022

Name: Key

SID:

There are 6 questions. Write clearly and show all of your work. You must justify all of your answers to receive credit. No calculators or notes are allowed.

You may use any of the following formulas from class:

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}\cos^{-1}(x)\,\mathrm{d}x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}\tan^{-1}(x)\,\mathrm{d}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

1. Evaluate the following integrals

(a) (10 pts)

$$\int \frac{x}{(x+1)(x+2)}\,\mathrm{d}x$$

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} = \frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x + (2A+B)}{(x+1)(x+2)}$$

so $A+B=1$
$2A+B=0$

$$\begin{array}{r} 2A+B=0 \\ -(A+B=1) \\ \hline A \quad = -1 \end{array} \rightarrow \text{So } B=2$$

$$\int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)}dx = -\ln|x+1| + 2\ln|x+2| + C$$

(b) (10 pts)

$$\int_0^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Improper integral since $\lim_{x\to 0^+} \frac{e^{\sqrt{x}}}{\sqrt{x}} = \infty$

$$= \lim_{R\to 0^+} \int_R^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

$u=\sqrt{x}$ if $x=2$ $u=\sqrt{2}$

$du = \frac{1}{2\sqrt{x}}dx$ if $x=R$ $u=\sqrt{R}$

$$= \lim_{R\to 0^+} \int_{\sqrt{R}}^{\sqrt{2}} 2e^u\,du = \lim_{R\to 0^+} 2e^u\Big|_{\sqrt{R}}^{\sqrt{2}} = \lim_{R\to 0^+} \left(2e^{\sqrt{2}} - 2e^{\sqrt{R}}\right)$$

$$= 2e^{\sqrt{2}} - 2e^0$$

2. (10 pts) Find the arclength of the function $f(x) = \frac{2}{3}x^{3/2}$ on the interval $[0, 2]$.

$$f'(x) = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = x^{1/2}$$

$$\text{arclength} = \int_0^2 \sqrt{1 + (f'(x))^2}\, dx = \int_0^2 \sqrt{1 + (\sqrt{x})^2}\, dx = \int_0^2 \sqrt{1+x}\, dx$$

$u = 1+x$ if $x = 0$ $u = 1$

$du = dx$ if $x = 2$ $u = 3$

$$= \int_1^3 \sqrt{u}\, du = \frac{2}{3} u^{3/2} \Big|_1^3 = \frac{2}{3}\left(3^{3/2} - 1^{3/2}\right)$$

3. (10 pts) Use the comparison test to determine if the following integral converges

$$\int_1^\infty \frac{1}{(\sqrt{x} + x^3)}\, dx$$

$\dfrac{1}{\sqrt{x} + x^3} \leq \dfrac{1}{x^3}$ for $x \geq 1$ since $\sqrt{x} > 0$

So since $\displaystyle\int_1^\infty \frac{1}{x^3}\, dx$ converges (p-integral with $p=3$)

$\displaystyle\int_1^\infty \frac{1}{\sqrt{x} + x^3}\, dx$ converges by the comparison test.

4. (10 pts) Determine the limit:

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+2}$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{2}{n^2}} = 1$$

Since $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

and $\lim\limits_{n\to\infty} \frac{2}{n^2} = 0$

5. (10 pts) Consider the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$$

(a) Show that the Nth partial sum is $S_N = 1 - \dfrac{1}{(N+1)^2}$

$$S_N = \left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{(N-1)^2}} - \cancel{\frac{1}{N^2}}\right) + \left(\cancel{\frac{1}{N^2}} - \frac{1}{(N+1)^2}\right)$$

$$= 1 - \frac{1}{(N+1)^2}$$

(b) Does $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$ converge?

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{(N+1)^2} \right) = 1$$

so the summation converges

6. True/False Circle the correct answer (2 pts each). You do not need to justify your answers.

(a) **True** / **False**: $\lim_{n\to\infty} (-1)^n$ converges. 1, -1, 1, -1, 1, -1, ...

(b) **True** / **False**: If $|r| < 1$, then $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$ this is the formula for a geometric series.

(c) **True** / **False**: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. this is the nth term divergence test

(d) **True** / **False**: If $0 \le f(x) \le g(x)$ and $\int_1^\infty f(x)\,\mathrm{d}x$ diverges, then $\int_1^\infty g(x)\,\mathrm{d}x$ diverges.

this is the comparison test for intograls

(e) **True** / **False**: If $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ diverges.

these p-intograls converge.