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Math 31B Dis 2D

Final Exam
Math 31B, Winter 2022

Midterm Exam: Tuesday, March 15.

Please read the policies for online exams posted on Canvas, under Modules – Syllabus and Policies. (The policies for this exam are the same as for the two midterms.)

You may solve these problems by any method you like (as long as the method is correct).

1. Evaluate
$$\frac{d}{dx}\left(2^{x^2+3x+2}\right).$$

2. Evaluate
$$\int \frac{e^x}{1+e^x}\,dx.$$

3. Evaluate
$$\int x^2 e^x\,dx.$$

4. Evaluate
$$\int_0^\infty \frac{dx}{x^2+5x+4}.$$
(You do not have to verify that the integral converges. I assure you that it does. Just find the value.)

5. Determine (with proof) whether the following integral converges or diverges:
$$\int_2^\infty \frac{x+3}{x^2+1}\,dx.$$

6. Determine (with proof) whether the following sum converges or diverges:

$$\sum_{n=2}^{\infty} \frac{n+3}{n^2+1}.$$

7. Determine (with proof) the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} e^{n^2} x^n.$$

8. Determine (with proof) the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2+n+1}.$$

9. Determine the Taylor series for

$$\frac{1}{x^2+3x+2}.$$

Hint: partial fractions. Find a partial fraction decomposition for $\frac{1}{x^2+3x+2}$, then use it to find the Taylor series.

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Math 31 B Final

1. $\frac{d}{dx}\left(2^{x^2+3x+2}\right)$

$\frac{d}{dt} 2^t = 2^t \ln(2)$ $\quad$ $\frac{d}{dx}\left(2^{x^2+3x+2}\right) = 2^{(x^2+3x+2)} \ln 2 \cdot (2x+3)$

$t = x^2 + 3x + 2$

2. $\int \frac{e^x}{1+e^x}\,dx$

$u = 1+e^x$, $du = e^x dx$

$\int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|1+e^x| + C$

3. $\int x^2 e^x\,dx$

$\int u\,dv = uv - \int v\,du$

$u = x^2 \quad dv = e^x dx$

$du = 2x\,dx \quad v = e^x$

$\Big\} \int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$

$\int 2xe^x dx$

$u = 2x \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$\Big\} \int 2xe^x dx = 2xe^x - \int 2e^x dx$

$= 2xe^x - 2\int e^x dx$

$= 2xe^x - 2e^x$

$\int x^2 e^x dx = x^2 e^x - (2xe^x - 2e^x) + C$

$= x^2 e^x - 2xe^x + 2e^x + C$

4. $\int_0^\infty \frac{dx}{x^2+5x+4} = \lim_{R\to\infty} \int_0^R \frac{1}{x^2+5x+4}\,dx$

$\frac{A}{(x+4)} + \frac{B}{(x+1)} = \frac{1}{x^2+5x+4}$

$A(x+1) + B(x+4) = 1$

if $x = -1$... $\quad$ if $x = -4$...

$3B = 1 \quad -3A = 1$

$B = 1/3 \quad A = -1/3$

$= \lim_{R\to\infty} \int_0^R \frac{-1}{3(x+4)} + \frac{1}{3(x+1)}\,dx$

$= \lim_{R\to\infty} \int_0^R \frac{1}{3(x+1)} - \frac{1}{3(x+4)}\,dx$

$= \lim_{R\to\infty} \frac{1}{3}\int_0^R \frac{1}{x+1} - \frac{1}{x+4}\,dx$

$= \lim_{R\to\infty} \frac{1}{3}\left[\ln|x+1| - \ln|x+4|\right]_0^R$

$= \lim_{R\to\infty} \left[\frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x+4|\right]_0^R$

$= \lim_{R\to\infty} \left[\frac{1}{3}\ln|R+1| - \frac{1}{3}\ln|R+4|\right] - \left[\frac{1}{3}\ln(1) - \frac{1}{3}\ln(4)\right]$ $\quad 0 \quad 0$

$= \frac{1}{3}\ln(4)$

5. $\int_2^\infty \frac{x+3}{x^2+1}\,dx = \lim_{R\to\infty}\int_2^R \frac{x+3}{x^2+1}\,dx = \lim_{R\to\infty}\int_2^R \frac{x}{x^2+1} + \frac{3}{x^2+1}\,dx = \lim_{R\to\infty} \underbrace{\int_2^R \frac{x}{x^2+1}\,dx}_{u=x^2+1 \quad du=2x\,dx} + \underbrace{3\int_2^R \frac{1}{x^2+1}\,dx}_{[3\tan^{-1}(x)]_2^R}$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int_2^R \frac{du}{u} = \frac{1}{2}[\ln|u|]_2^R$

$= \frac{1}{2}[\ln|x^2+1|]_2^R$

$= \lim_{R\to\infty}\left[\frac{1}{2}\ln|x^2+1| + 3\tan^{-1}(x)\right]_2^R$

$= \lim_{R\to\infty}\left(\frac{1}{2}\ln|R^2+1| + 3\tan^{-1}(R)\right) - \left(\frac{1}{2}\ln 5 + 3\tan^{-1}(2)\right)$ ∞

$= \infty$ DNE

By the improper integrals test, $\int_2^\infty \frac{x+3}{x^2+1}\,dx \approx \lim_{R\to\infty}\int_2^R \frac{x+3}{x^2+1}\,dx$. Since the limit does not exist, the improper integral $\int_2^\infty \frac{x+3}{x^2+1}\,dx$ diverges.

6. $\sum_{n=2}^{\infty} \frac{n+3}{n^2+1}$

Integral Test:

Check conditions: $f(x) = \frac{n+3}{n^2+1}$ is a decreasing function such that $f(x) \geq 0$ for all $x \geq 2$

Apply the Integral Test:

$\sum_{n=2}^{\infty} \frac{n+3}{n^2+1} = \int_2^\infty \frac{n+3}{n^2+1}\,dx = \lim_{R\to\infty}\int_2^R \frac{n+3}{n^2+1}\,dx$

According to my work in #5, $\int_2^\infty \frac{n+3}{n^2+1}\,dx$ diverges, so by the Integral Test, $\sum_{n=2}^{\infty} \frac{n+3}{n^2+1}$ must also diverge.

7. $\sum_{n=1}^{\infty} e^{n^2} x^n$

Ratio Test:

x is a constant

$\lim_{n\to\infty}\left|\frac{e^{(n+1)^2} x^{n+1}}{e^{n^2} x^n}\right| = \lim_{n\to\infty}\left|\frac{e^{n^2+2n+1} x^{n+1}}{e^{n^2} x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{e^{n^2}} e^{2n+1} \cancel{x^n} x^1}{\cancel{e^{n^2}} \cancel{x^n}}\right| = \lim_{n\to\infty} e^{2n+1} x = (x)(\infty)$

By the Ratio Test, the power series diverges because $\lim_{n\to\infty}\left|\frac{e^{(n+1)^2} x^{n+1}}{e^n x^n}\right| > 1$.
However, the definition of a power series states that all power series converge at $x=0$.
Therefore, the radius of convergence is 0 because $\sum_{n=1}^{\infty} e^{n^2} x^n$ only converges for $x=0$.

8. $\sum_{n=1}^{\infty} \frac{x^n}{n^2+n+1}$

Ratio Test:

$\lim_{n\to\infty}\left|\frac{\frac{x^{(n+1)}}{(n+1)^2+(n+1)+1}}{\frac{x^n}{n^2+n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n} x^1}{(n+1)^2+n+2} \cdot \frac{n^2+n+1}{\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{x(n^2+n+1)}{(n+1)^2+n+2}\right| = |x| \lim_{n\to\infty}\left|\frac{n^2+n+1}{n^2+2n+1+n+2}\right|$

$= |x| \lim_{n\to\infty}\left|\frac{n^2+n+1}{n^2+3n+3}\right| = |x|(1) = |x|$

$\lim_{n\to\infty}\left|\frac{n^2+n+1}{n^2+3n+3}\right| = \frac{\infty}{\infty}$

$\mathcal{L} \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right| = \frac{\infty}{\infty}$

$\mathcal{L} \lim_{n\to\infty}\left|\frac{2}{2}\right| = 1$

By the Ratio Test, the series converges when $|x|<1$ and diverges when $|x|>1$, so the Radius of Convergence $=1$.

Check Conditions for L'Hôpital:

1. Numerator and denominator are differentiable
2. indeterminate form $\frac{\infty}{\infty}$
3. derivative of denominator is non-zero when $n \approx 0$

9. $$\frac{1}{x^2+3x+2} = \underbrace{\frac{A}{(x+2)} + \frac{B}{(x+1)}}_{\substack{A(x+1)+B(x+2)=1 \\ \text{if } x=-1\ldots \quad \text{if } x=-2\ldots \\ B=1 \quad -A=1 \\ A=-1}} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

Geometric Series

$$\frac{1}{x+2} = \frac{1}{2(\frac{1}{2}x+1)} = \frac{1}{2(1-(-\frac{1}{2}x))} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n$$

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$= \sum_{n=0}^{\infty}(-x)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n$$