Midterm 2
Math 31B, Winter 2022

Midterm Exam: Friday, February 25.

Please read the policies for online midterms posted on Canvas, under Modules – Syllabus and Policies.

You may solve these problems by any method you like (as long as the method is correct).

1. Determine whether the following limit exists, and if so, evaluate it:
$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

2. Determine whether the following integral converges, and if so, evaluate it:
$$\int_5^\infty \frac{dx}{x^2-1}.$$

3. Determine whether the following limit exists, and if so, evaluate it:
$$\lim_{n\to\infty} \sqrt{n^2+1} - 2n.$$

4. Determine whether the following sum converges, and if so, evaluate it:
$$\sum_{n=0}^{\infty} 3^{1-2n} + 2^{2-n}.$$

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Math 31B Dis 2D
Midterm 2

Check Conditions for L'Hôpital:

1. Numerator and denominator are differentiable
2. indeterminate form $\frac{0}{0}$
3. derivative of denominator is non-zero when $x \approx 0$

1. $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{0}{0}$

$\mathcal{L}$ $\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0}$

$\mathcal{L}$ $\lim_{x \to 0} \frac{-\sin x}{6x} = \frac{0}{0}$

$\mathcal{L}$ $\lim_{x \to 0} \frac{-\cos x}{6} = \frac{-1}{6}$

2. $\int_5^\infty \frac{dx}{x^2-1} = \lim_{R\to\infty} \underbrace{\int_5^R \frac{dx}{x^2-1}}$ $= \lim_{R\to\infty} \int_5^R \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx = \lim_{R\to\infty} \int_5^R \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\,dx = \lim_{R\to\infty} \frac{1}{2}\int_5^R \frac{1}{x-1} - \frac{1}{x+1}\,dx$

$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$A(x+1) + B(x-1) = 1$

if $x=-1$...   if $x=1$...

$-2B = 1$ $2A = 1$

$B = \frac{-1}{2}$ $A = \frac{1}{2}$

$= \lim_{R\to\infty} \frac{1}{2}\left[\ln|x-1| - \ln|x+1|\right]_5^R$

$= \lim_{R\to\infty} \left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_5^R$

$= \lim_{R\to\infty} \frac{1}{2}\ln\left|\cancel{\frac{R-1}{R+1}}\right| \xrightarrow{} 0 - \frac{1}{2}\ln\left|\frac{4}{6}\right|$

$= -\frac{1}{2}\ln\left|\frac{2}{3}\right|$ converges

3. $\lim_{n\to\infty} \sqrt{n^2+1} - 2n$

$= \lim_{n\to\infty} \frac{\sqrt{n^2+1} - 2n}{1} \times \frac{\sqrt{n^2+1} + 2n}{\sqrt{n^2+1} + 2n}$

$= \lim_{n\to\infty} \frac{(n^2+1) - 4n^2}{\sqrt{n^2+1} + 2n} = \lim_{n\to\infty} \frac{-3n^2+1}{\sqrt{n^2+1}+2n} = \lim_{n\to\infty} \frac{\cancel{n}(-3n+\frac{1}{n})}{\cancel{n}(\sqrt{1+\frac{1}{n^2}}+2)} = \lim_{n\to\infty} \frac{\overset{\to -\infty}{-3n} + \cancel{\frac{1}{n}}^{\to 0}}{\sqrt{1+\cancel{\frac{1}{n^2}}^{\to 0}}+2} = \frac{-\infty}{3} = -\infty$

4 $\sum_{n=0}^\infty 3^{1-2n} + 2^{2-n}$

$= \sum_{n=0}^\infty 3^{1-2n} + \sum_{n=0}^\infty 2^{2-n}$

$\sum_{n=0}^\infty 3^1 \cdot 3^{-2n} = 3\sum_{n=0}^\infty \left(\frac{1}{9}\right)^n$

By the geometric series formula $\sum_{n=0}^\infty 3^{1-2n} = 3\left(\frac{1}{1-1/9}\right) = 3 \times \frac{9}{8} = \frac{27}{8}$
The sum converges because $|\frac{1}{9}| < 1$ and the value of the sum is $27/8$.

$\sum_{n=0}^\infty 2^{2-n} = \sum_{n=0}^\infty 2^2 \cdot 2^{-n} = 4\sum_{n=0}^\infty \left(\frac{1}{2}\right)^n$

By the geometric series formula $\sum_{n=0}^\infty 2^{2-n} = 4\left(\frac{1}{1-1/2}\right) = 4 \times 2 = 8$
The sum converges because $|\frac{1}{2}| < 1$ and the value of the sum is 8.

$\sum_{n=0}^\infty 3^{1-2n} + 2^{2-n} = \sum_{n=0}^\infty 3^{1-2n} + \sum_{n=0}^\infty 2^{2-n}$

$= \frac{27}{8} + 8$

$= \frac{27}{8} + \frac{64}{8} = \frac{91}{8}$

converges

If we have two convergent series, then their sum must also converge at the sum of their respective values.