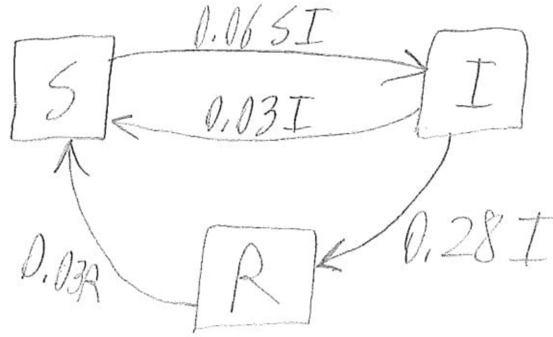
1. (10 points) The flu is spreading in your dorm. Set up a discrete-time model based on the assumptions below. If your model is linear, write down its matrix. If not, explain why not.

- The population of the dorm is divided into residents who are **susceptible**, those who are currently **infected**, and those who have **recovered** with immunity.
- Susceptible people become infected when they come into contact with an infected person. Use a proportionality constant of 0.06 for this.
- When infected people get well, they usually have some immunity to the disease, but a small fraction do not. So assume that each day, 28% of infected people recover with immunity, but an additional 3% of infected people go right back to being susceptible again.
- Immunity does not last forever, so each day 3% of those who have recovered become susceptible again.

$S$ = # of susceptible residents
$I$ = # of infected residents
$R$ = # of recovered residents (with immunity)




$$\begin{cases} S_{t+1} - S_t = 0.03 I_t + 0.03 R_t - 0.06 S_t I_t \\ I_{t+1} - I_t = 0.06 S_t I_t - 0.03 I_t - 0.28 I_t \\ R_{t+1} - R_t = 0.28 I_t - 0.03 R_t \end{cases}$$

$$\boxed{\begin{cases} S_{t+1} = S_t + 0.03 I_t + 0.03 R_t - 0.06 S_t I_t \\ I_{t+1} = 0.69 I_t + 0.06 S_t I_t \\ R_{t+1} = 0.28 I_t + 0.97 R_t \end{cases}}$$

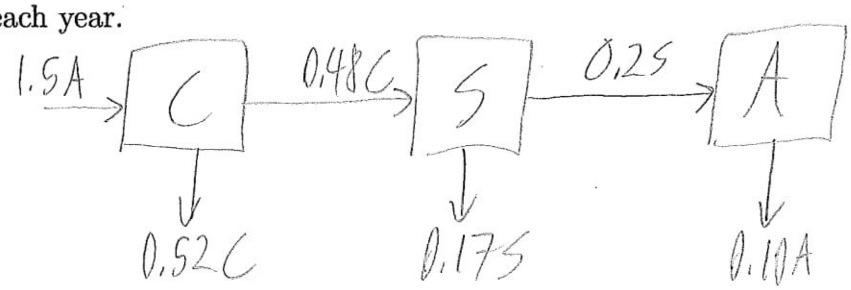
This model is not linear because of these two terms.

2. (10 points) The Siberian tiger is an endangered subspecies of tiger that inhabits forests in Siberia and northern China. Set up a discrete-time model for a Siberian tiger population based on the assumptions below. If your model is linear, write down its matrix. If not, explain why not.

- The population is divided into cubs (less than a year old), subadults (1–3 years old) and adults.
- On average, adults have 1.5 cubs per year.
- Siberian tiger cubs only remain cubs for one year. 52% of cubs die during this first year.
- On average, 63% of subadults survive as subadults from one year to the next.
- 20% of subadults mature into adults each year.
- 10% of adults die each year.

83% survive, so 17% die

$C$ = # of cubs
$S$ = # of sub-adults
$A$ = # of adults




$$\begin{cases} C_{t+1} - C_t = 1.5A_t - 0.52C_t - 0.48C_t \\ S_{t+1} - S_t = 0.48C_t - 0.17S_t - 0.20S_t \\ A_{t+1} - A_t = 0.20S_t - 0.10A_t \end{cases}$$

$$\Downarrow$$

$$\begin{cases} C_{t+1} = \cancel{C_t} - \cancel{0.52C_t} - \cancel{0.48C_t} + 1.5A_t = & 1.5A_t \\ S_{t+1} = 0.48C_t + S_t - 0.17S_t - 0.20S_t = & 0.48C_t + 0.63S_t \\ A_{t+1} = A_t - 0.10A_t + 0.20S_t = & 0.20S_t + 0.90A_t \end{cases}$$

In matrix form:

$$\boxed{\begin{bmatrix} C_{t+1} \\ S_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.5 \\ 0.48 & 0.63 & 0 \\ 0 & 0.20 & 0.90 \end{bmatrix} \begin{bmatrix} C_t \\ S_t \\ A_t \end{bmatrix}}$$

3. The parts of this problem are independent of each other.

(c) (4 points) Suppose $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a function for which

$$g\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}5\\2\\-1\end{bmatrix}, \quad g\left(\begin{bmatrix}-4\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\3\\3\end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = \begin{bmatrix}3\\6\\2\end{bmatrix}.$$

Could $g$ be a linear function? Why or why not?

Notice that $\begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$. So <u>if</u> $g$ is linear, then

$$g\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = g\left(\begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}-4\\2\end{bmatrix}\right) = g\left(\begin{bmatrix}5\\2\end{bmatrix}\right) + g\left(\begin{bmatrix}-4\\2\end{bmatrix}\right) = \begin{bmatrix}5\\2\\-1\end{bmatrix} + \begin{bmatrix}-2\\3\\3\end{bmatrix} = \begin{bmatrix}3\\5\\2\end{bmatrix}.$$

But the problem says that $g\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = \begin{bmatrix}3\\6\\2\end{bmatrix}$, and $\begin{bmatrix}3\\6\\2\end{bmatrix} \neq \begin{bmatrix}3\\5\\2\end{bmatrix}$.

Therefore <u>$g$ cannot be linear.</u>

4. Let $\mathbf{u}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. These two vectors form a basis for $\mathbb{R}^2$, which defines a new coordinate system that we'll call $R, S$-coordinates for the remainder of this problem.

(a) (4 points) Find the $R, S$-coordinates for the vector $\begin{bmatrix} -10 \\ 13 \end{bmatrix}$.

$$\begin{bmatrix} -10 \\ 13 \end{bmatrix} = R\begin{bmatrix} 4 \\ -1 \end{bmatrix} + S\begin{bmatrix} 2 \\ 3 \end{bmatrix} \Longleftrightarrow \begin{cases} -10 = 4R + 2S \\ 13 = -R + 3S \end{cases}$$

$R = 3S - 13$, so $-10 = 4(3S - 13) + 2S = 12S - 52 + 2S$

$14S = 42$

$S = 3$

$R = 3 \cdot 3 - 13 = -4$

$\boxed{R,S\text{-coordinates: } R = -4,\ S = 3}$

(b) (4 points) Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear function for which

$$f\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}.$$

Use your answer to part (a) and the *definition of linear functions* to compute $f\left(\begin{bmatrix} -10 \\ 13 \end{bmatrix}\right)$.

From part (a): $\begin{bmatrix} -10 \\ 13 \end{bmatrix} = -4\begin{bmatrix} 4 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

So $f\left(\begin{bmatrix} -10 \\ 13 \end{bmatrix}\right) = f\left(-4\begin{bmatrix} 4 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = f\left(-4\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) + f\left(3\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

$$= -4 \cdot f\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) + 3 \cdot f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

$$= -4\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -24 \\ -17 \\ 17 \end{bmatrix}}$$

5. The following matrix $M$ describes a discrete-time model of dugongs (a marine mammal native to the Indian and southern Pacific oceans, closely related to manatees), in which the population has been subdivided into calves, juveniles, and adults (in that order).

$$M = \begin{bmatrix} 0.3 & 0 & 0.25 \\ 0.3 & 0.6 & 0 \\ 0 & 0.12 & 0.93 \end{bmatrix}$$

(a) (5 points) Explain what each nonzero number in this matrix tells you. What are the per-capita death rates for each life stage?

- Each year, 30% of calves remain calves, 30% of calves mature into juveniles, and the remaining 40% of calves die.
- Each year, 60% of juveniles remain as juveniles, 12% of juveniles mature into adults, and the remaining 28% of juveniles die.
- Each year, 93% of adults survive, and the other 7% of adults die.
- On average, each adult gives birth to 0.25 calves each year.

Question 5 continues on the next page...

(b) (4 points) Suppose that the matrix $M$ above describes what happens to this population in a typical year, but the matrix for a La Niña year is

$$L = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.25 & 0.5 & 0 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

Compute the matrix that represents a La Niña year followed by a typical year.

$$ML = \begin{bmatrix} 0.3 & 0 & 0.25 \\ 0.3 & 0.6 & 0 \\ 0 & 0.12 & 0.93 \end{bmatrix} \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.25 & 0.5 & 0 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 0.06 & 0.025 & 0.285 \\ 0.21 & 0.3 & 0.06 \\ 0.03 & 0.153 & 0.837 \end{bmatrix}}$$

(c) (2 points) Write down an expression for (but do not actually compute) the matrix that would represent a sequence of four typical years, followed by a La Niña year, another typical year, and two more La Niña years.

$$\boxed{LLMLMMMM} = \boxed{L^2MLM^4}$$