LS 30B-3 Midterm 2

Elizabeth Frances Elton

TOTAL POINTS

74 / 78

QUESTION 1

Heartbeat Poincare Plot 8 pts

1.1 Create Poincare Plot **4 / 4**

✓ **- 0 pts** Correct

1.2 Plot Tells You **4 / 4**

✓ **- 0 pts** Correct

QUESTION 2

Characteristics of Chaos 10 pts

2.1 Identify & Describe **8 / 10**

✓ **- 1.5 pts** Explanation(Deterministic) missing/incorrect

✓ **- 0.5 pts** Explanation(Sensitive dependence on initial conditions) partially incorrect/incomplete

2.2 Characteristics Shared with Stable EP **0 / 0**

✓ **- 0 pts** Not graded

2.3 Characteristics Shared with Limit Cycle Attractor **0 / 0**

✓ **- 0 pts** Not graded

QUESTION 3

Modeling Shark Population 15 pts

3.1 Write Matrix Model **8 / 8**

✓ **- 0 pts** Correct

3.2 Population Growth **3 / 4**

✓ **- 1 pts** Wrong growing percentage

3.3 Number of Juveniles **3 / 3**

✓ **- 0 pts** Correct

QUESTION 4

Interpreting & Analyzing Flu Model 19 pts

4.1 Biological Meaning **3 / 4**

✓ **- 1 pts** wrong explanation for 0.3

4.2 Lethal **2 / 2**

✓ **- 0 pts** Correct

4.3 Modify Model for Slower Recovery **2 / 2**

✓ **- 0 pts** Correct

4.4 Number of Susceptible and Infected on Next Day **3 / 3**

✓ **- 0 pts** Correct

4.5 Calculate Eigenvalues **4 / 4**

✓ **- 0 pts** Correct (1, 0.5)

4.6 Calculate Eigenvectors **4 / 4**

✓ **- 0 pts** Correct

QUESTION 5

Linear Functions 9 pts

5.1 Could Be a Linear Function **3 / 3**

✓ **- 0 pts** Correct

5.2 Find Matrix Representing f **3 / 3**

✓ **- 0 pts** Correct

5.3 Linear Combination & Linear Function **3 / 3**

✓ **- 0 pts** Correct

QUESTION 6

Matching Eigenvalues and Trajectories 8 pts

6.1 Saddle Point **2 / 2**

✓ **- 0 pts** Correct

6.2 Unstable Spiral **2 / 2**

✓ **- 0 pts** Correct

6.3 Stable Spiral **2 / 2**

✓ **- 0 pts** Correct

6.4 Unstable Node with Oscillations **2 / 2**

✓ **- 0 pts** **Correct**

QUESTION 7

Google PageRank 9 pts

7.1 "Points-to" Matrix **2 / 2**

✓ **- 0 pts** **Correct**

7.2 "Links-to" Matrix **2 / 2**

✓ **- 0 pts** **Correct**

7.3 Initial Condition Vector of Equal Weights

1 / 1

✓ **- 0 pts** **Correct**

7.4 Highest Page Rank **4 / 4**

✓ **- 0 pts** **Correct**

A. Garfinkel LS 30B-3 February 27, 2018

Midterm 2 Exam (Version A)

First Name: Elizabeth

Last Name: Elton

Last 6 digits of UID: 986788

Section # (TA): Ming (3C?)

By signing below, you confirm that you did not cheat on this exam. No exam without a signature will be graded.
Signature:

Instructions: Do not open this exam until instructed to do so. You will have 1 hour and 50 minutes to complete the exam. Please print the last 6 digits of your student ID number above and on each page of the exam. You may not use books, notes, or any other material to help you. You may use a scientific or 4-function calculator for this exam. Please make sure your phone is silenced and stowed at the front of the room. Please write only on the space below each problem. We are providing plenty of space for each problem, so you should not need additional space or tiny handwriting to answer any of the problems. However, you can use pages 13 and 14 (or request additional paper) as scratch paper which will not be graded. After you start the exam, tear off pages 13 and 14 and turn them in separately.

Please do not write below this line.

Problem	Max	Score
1	8	
2	20	
3	15	
4	19	
5	9	
6	8	
7	9	
Total	88	

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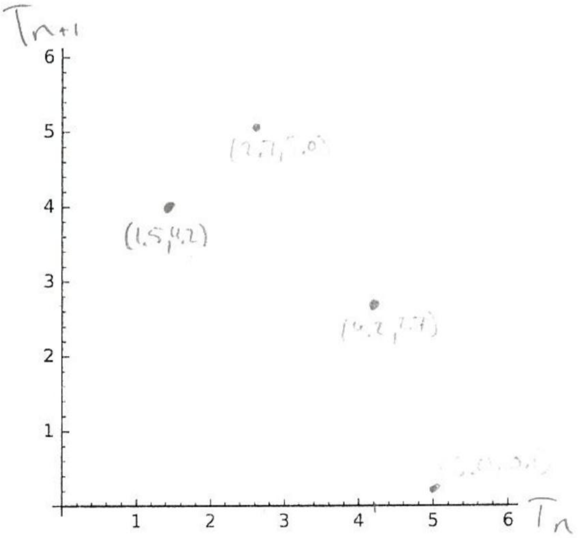
1. For a heart disease patient, a continuous electrocardiogram is used to monitor the patient's heart. A preliminary analysis of the data's time series shows highly irregular behavior. To reduce the data to a discrete-time dataset, you measure the time between the heartbeats: let T_n be the time (in seconds) between the n-th heartbeat and the one after it.

   a. (4 points) Suppose $T_1 = 1.5, T_2 = 4.2, T_3 = 2.7, T_4 = 5.0$, and $T_5 = 0.1$. Create a Poincaré plot of these values on the axes provided. Label the axes appropriately and label each point with its coordinates.



   b. (4 points) After monitoring a different patient's heart for several hours, you have accumulated a new list with a few thousand heartbeat intervals (which also have a highly irregular time series), and you create a Poincaré plot using all of them. What could this plot tell you about the mathematical behavior of this patient's heartbeats?

The plot can tell you whether the system is chaotic or random. If the plot forms a regular curve, like: then the

behavior is chaotic. However, if it just forms a general blob of points:

then the system is random.

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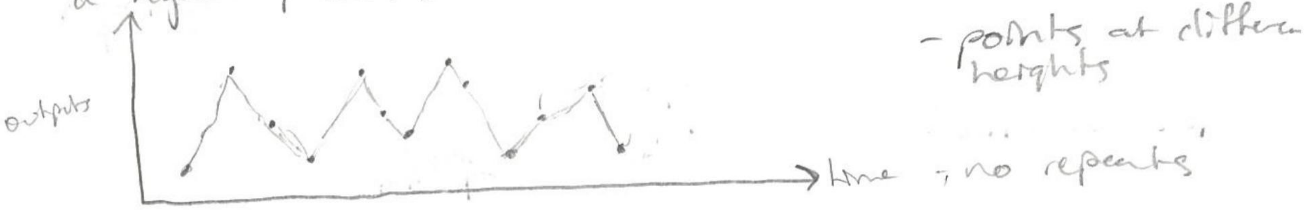
2.

a. (10 points) Identify and briefly describe at least 4 characteristics of chaos for a system of differential equations or a discrete-time dynamical system. You are welcome to use examples, equations, graphs, and/or diagrams in addition to your verbal descriptions. To receive full credit, you must show that you understand what the keywords mean, not just give the keywords.

Bounded: Chaotic systems are bounded, meaning there is a definite range of values that the output of the system can be, and it cannot stray outside these values (ie, it is not infinite).

Deterministic: Chaotic systems are deterministic in that we can determine from them an estimate for the long term behavior. This is because chaotic systems have whats called a strange attractor that each system appears to come close to (but not completely) overlap

Aperiodic: Chaotic systems are aperiodic in that no poi (output) is ever repeated, so the entire system doesn't have a regular pattern.



Sensitive Dependence on Initial Conditions:

Chaotic systems have sensitive dependence on initial conditions, meaning that the behavior of one system will be completely different than the behavior of another system at a different starting point.

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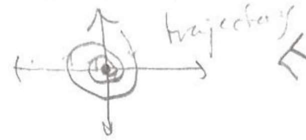
b. (5 points) Which features are shared with a system that is governed by a stable equilibrium point? Provide evidence to support your response for each feature using appropriate equations, graphs, and/or diagrams.

Bounded: Systems with stable equilibrium points have a time series with damped oscillations, in which all points will eventually converge to the bounds of the stable equilibrium

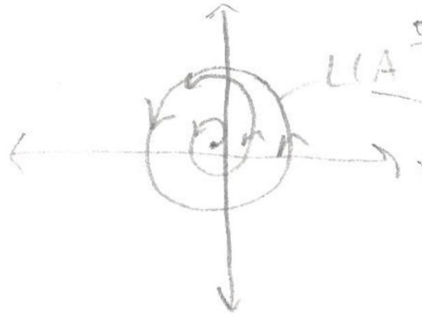


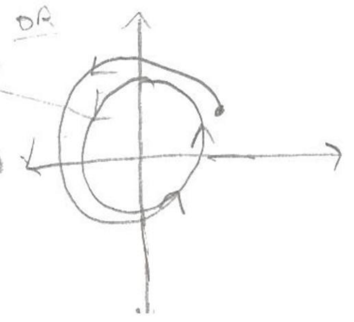
Deterministic:
Systems with stable equilibriums also have a trajectory with an inward moving spiral. Because the equilibrium is stable, we know the long-term behavior of all [illegible] (to go towards [illegible] end spot) the equilibrium point, so it is therefore deterministic.

c. (5 points) Which features are shared with a system that is governed by a limit cycle attractor? Provide evidence to support your response for each feature using appropriate equations, graphs, and/or diagrams.

Bounded: Systems with a limit cycle attractor are bounded by the maximum and minimum values of the stable oscillations, and it cannot [illegible]

Deterministic: Systems with a limit cycle attractor will all eventually converge on the attractor and follow that pattern, showing how they are deterministic.





- either starting point, we know the long term behavior

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3. Great white sharks (*Carcharodon carcharias*) are famed for their ferocity worldwide and are found in increasing numbers off the beaches of Southern California. We will model only female sharks with three life stages: pups, juveniles, and adults.
   a. (8 points) Write a discrete-time matrix model of a great white shark population using the following assumptions. Make sure you identify each state variable.
      - Each adult shark has approximately 6 pups per litter. — assume this per year
      - Sharks can only be pups for one year. — 100% pups → juv.?
      - Juveniles take 10 years to mature into adults, so 10% of juveniles become adults each year.
      - 8% of pups survive each year. — to become pups
      - 22% of juveniles survive each year and either become adults or stay juveniles.
      - 13% of adults die each year.

      survive: 10% → Ad, 12% stay juv + survive

      Your final answer should look something like

      (next state) = M(current state)

      (where M is a matrix and appropriate mathematical expressions in place of the words).

$$\Delta P = 6A - 0.08P - 0.92P$$
$$\Delta J = 0.08P - 0.88J$$
$$\Delta A = 0.1J - 0.13A$$

+P, +J, +A

P = # of pups
J = # of juveniles
A = # of adults

$$P_{n+1} = 6A_n$$
$$J_{n+1} = 0.08P_n + 0.12J_n$$
$$A_{n+1} = 0.1J_n + 0.87A_n$$

$$\begin{pmatrix} P_{n+1} \\ J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 6 \\ 0.08 & 0.12 & 0 \\ 0 & 0.1 & 0.87 \end{bmatrix} \begin{pmatrix} P_n \\ J_n \\ A_n \end{pmatrix}$$

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b. (4 points) Suppose the matrix you wrote in part (a) had the following (approximate) eigenvalues and eigenvectors:

- $\begin{pmatrix} 0.999 \\ 0.139 \\ 0.018 \end{pmatrix}$ with eigenvalue 0.111
- $\begin{pmatrix} 0.959 \\ 0.231 \\ 0.161 \end{pmatrix}$ with eigenvalue 1.013 → Dominant

principle

- $\begin{pmatrix} -0.960 \\ 0.300 \\ -0.068 \end{pmatrix}$ with eigenvalue 0.426

Do you expect the population to persist or go extinct in the long run? How much is the population growing or declining each year? Explain your answers using the information in this problem.

The population should persist in the long run since the dominant eigenvalue (the one with the largest absolute value) is greater than 1. The population each year will be 101.3% larger than the population in the former year, meaning the growth rate is 1.3% per year.

c. (3 points) Suppose that at some point in the distant future, the population has 100 adult white sharks. Approximately how many juvenile white sharks would you expect there to be at that time? Explain or show how you calculate this using the information in this problem.

Principle eigenvector → $\begin{pmatrix} 0.959 \\ 0.231 \\ 0.161 \end{pmatrix} \begin{pmatrix} P_n \\ J_n \\ A_n \end{pmatrix}$ - The principle eigenvector provides the ratio of the life stages to one another.

Therefore, $\dfrac{0.231}{0.161} = \dfrac{x}{100}$ x = number of juveniles

$= \boxed{143.48 \text{ juveniles}}$

(or 143 since you cannot have 0.48 of a shark)

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4. The discrete-time matrix model below describes the spread of flu in UCLA's residence halls this winter. S_n is the number of susceptible people on day n. I_n is the number of infected people on day n, as recorded by the Ashe Center each day.

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix}$$

a. (4 points) Briefly explain the biological meaning of each element in the matrix. One sentence per element is sufficient.

0.8 → 80% of susceptible people ~~who~~ stay susceptible per d
0.3 → 30% of infected people ~~who~~ become infected p
0.2 → 20% of susceptible people ~~who~~ become infec
0.7 → 70% of infected people ~~who~~ stay infected
pe

b. (2 points) Is this disease lethal? (In other words, does this disease cause death?) Support your answer using information provided in this problem.

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ 0.8 & 0.3 \\ +0.2 & +0.7 \\ \hline 1 & 1 \end{array}$$

Since the columns of the matrix add up to 1, the matrix is a closed system meaning that individuals can only be susceptible or infected, not dead.

c. (2 points) During final exams, everyone is very stressed out, so they recover from the infection much more slowly. Modify the matrix model to reflect this change.

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{bmatrix} 0.8 & 0.01 \\ 0.2 & 0.99 \end{bmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix}$$

only 1% of infected people become susceptible (recover) per day
the rest stay infected

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$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix}$$

d. (3 points) If your floor had 120 susceptible people and 10 infected people on January 7 when everyone returned from winter vacation, how many people should have been susceptible and infected on January 8 according to the original model (directly above)? Make sure you show your work.

$S_{n+1} = 0.8(120) + 0.3(10) = 96 + 3 = \underline{99}$ $\underline{\text{susceptible}}$

$I_{n+1} = 0.2(120) + 0.7(10) = 24 + 7 = \underline{31}$ $\underline{\text{infected}}$

e. (4 points) Calculate the eigenvalues for the original matrix (at the top of this page). You do not need to interpret them. Make sure you show your work.

$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ $\lambda = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

$\lambda = \dfrac{1.5 \pm \sqrt{2.25 - 4(0.56 - 0.06)}}{2}$

$\lambda = \dfrac{1.5 \pm 0.5}{2}$ $\boxed{\lambda_1 = 1 \quad \lambda_2 = 0.5}$

f. (4 points) Suppose another strain of the flu has the discrete-time matrix model:

$$\begin{pmatrix} S_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{pmatrix} S_n \\ I_n \end{pmatrix}$$

The eigenvalues of this new matrix are 1 and 0.4. What are the eigenvectors corresponding to each eigenvalue? You should give two digits after the decimal point (e.g., 6.55). Make sure you show your work.

$\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1v_1 \\ 1v_2 \end{pmatrix}$

$0.9v_1 + 0.5v_2 = 1v_1$

$0.5v_2 = 0.1v_1$

$v_2 = 0.20v_1$

when $\lambda = 1$ $\begin{pmatrix} 1 \\ 0.20 \end{pmatrix}$

$\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0.4v_1 \\ 0.4v_2 \end{pmatrix}$

$0.9v_1 + 0.5v_2 = 0.4v_1$

$0.5v_2 = -0.5v_1$

$v_2 = -1v_1$

when $\lambda = 0.4$ $\begin{pmatrix} 1.00 \\ -1.00 \end{pmatrix}$

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5. The three parts of this question are independent of each other. For each part, make sure you explain or show your work.

a. (3 points) Suppose that $g: \mathbb{R}^3 \to \mathbb{R}^2$ is a function for which

$$g\left(\begin{pmatrix}1\\1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\3\end{pmatrix},\ g\left(\begin{pmatrix}2\\3\\1\end{pmatrix}\right) = \begin{pmatrix}3\\4\end{pmatrix},\ \text{and } g\left(\begin{pmatrix}3\\4\\3\end{pmatrix}\right) = \begin{pmatrix}5\\5\end{pmatrix}$$

Could g be a linear function? Why or why not?

linear definition: $f(x+a) = f(x) + f(a)$

$\therefore\ g\left(\begin{pmatrix}3\\4\\3\end{pmatrix}\right) \overset{\text{should}}{=} g\left(\begin{pmatrix}2\\3\\1\end{pmatrix}\right) + g\left(\begin{pmatrix}1\\1\\2\end{pmatrix}\right)$

$\begin{pmatrix}5\\5\end{pmatrix} \overset{\text{should}}{=} \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix}$

$\begin{pmatrix}5\\5\end{pmatrix} \neq \begin{pmatrix}5\\7\end{pmatrix} \rightarrow$ not linear $\rightarrow$ doesn't follow the linear definition

b. (3 points) The function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If $f\left(\begin{pmatrix}3\\0\end{pmatrix}\right) = \begin{pmatrix}9\\3\end{pmatrix}$ and $f\left(\begin{pmatrix}0\\2\end{pmatrix}\right) = \begin{pmatrix}4\\2\end{pmatrix}$, find the matrix representing f.

$f\left(\begin{pmatrix}3\\0\end{pmatrix}\right) = \begin{pmatrix}9\\3\end{pmatrix} \therefore f\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}3\\1\end{pmatrix}$

$f\left(\begin{pmatrix}0\\2\end{pmatrix}\right) = \begin{pmatrix}4\\2\end{pmatrix} \therefore f\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}2\\1\end{pmatrix}$

$f = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$

c. (3 points) The function $h: \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If $h\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}3\\1\end{pmatrix}$ and $h\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}2\\-1\end{pmatrix}$, what is $h\left(\begin{pmatrix}5\\6\end{pmatrix}\right)$?

$h\left(\begin{pmatrix}5\\6\end{pmatrix}\right) = 3 \cdot h\left(\begin{pmatrix}1\\2\end{pmatrix}\right) + h\left(\begin{pmatrix}2\\0\end{pmatrix}\right)$

$h\left(\begin{pmatrix}5\\6\end{pmatrix}\right) = \begin{pmatrix}9\\3\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}11\\2\end{pmatrix}$

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6. (8 points) We can explore Romeo's and Juliet's feelings for each other using the matrix model:

$$\begin{pmatrix} R_{n+1} \\ J_{n+1} \end{pmatrix} = M \begin{pmatrix} R_n \\ J_n \end{pmatrix},$$

where M can be a variety of 2x2 matrices. Write the number of the trajectory graph (on page 13) that corresponds to each matrix and its eigenvalues and state your reasoning. Note: all trajectories start with initial condition at the large dot at (1,1) and are based on at least 5 iterations. Hint: the absolute value of the complex number $a + bi$, written as $|a + bi|$, is equal to $\sqrt{a^2 + b^2}$.

Matrix, Eigenvalues, and Matching Trajectory	Reasoning
a. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\lambda_1 = 3.73$ $\lambda_2 = 0.27$ Trajectory: V	- λ_1 is stretching and λ_2 is shrinking, but λ_1 is much larger than λ_2, so the effect of λ_2 [illegible] be visible [illegible] a few iterations
b. $B = \begin{bmatrix} 1.2 & -1 \\ 0.9 & 0.2 \end{bmatrix}$ $\lambda_1 = 0.70 + 0.81i$ $\lambda_2 = 0.70 - 0.81i$ Trajectory: iv	$\lvert\lambda_1\rvert = \sqrt{.7^2+.81^2} = \sqrt{1.4161} = 1.19$ - imagnary means rotating, $\lvert\lambda\rvert > 1$ means growing
c. $C = \begin{bmatrix} 0.1 & 0.3 \\ -1 & 0.2 \end{bmatrix}$ $\lambda_1 = 0.15 + 0.55i$ $\lambda_2 = 0.15 - 0.55i$ Trajectory: i	$\lvert\lambda_1\rvert = \sqrt{0.15^2+0.55^2} = \sqrt{0.325} = 0.570$ - imagnary means rotating, $\lvert\lambda\rvert < 1$ means shrinking
d. $D = \begin{bmatrix} 0.1 & 4 \\ 0.4 & 0.2 \end{bmatrix}$ $\lambda_1 = -1.12$ $\lambda_2 = 1.42$ Trajectory: ii	- both are stretching, but λ_1 is stretching and flipping

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7. Suppose we have a network of three webpages, X, Y, and Z, with links to other pages as shown below. As part of your interview for a job at Google, you are given the task of building the Google PageRank vector for this simple network. Note that some information for the example in the textbook and in class is shown on the last page of this exam for your reference.

Page X	Page Y	Page Z
Page Y Page Z	Page X	Page X Page Y

a. (2 points) Write the "points-to" matrix P.

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

b. (2 points) Write the "links-to" matrix L.

$$L = \begin{bmatrix} 0 & 1 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{bmatrix}$$

c. (1 point) Write an initial condition PR_0 that is the vector of equal weights to each page.

3 pages = each one gets 1/3 weight

$$PR_0 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

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d. (4 points) Suppose the eigenvalues and corresponding eigenvectors of the "links-to" matrix for another set of pages K,L,M are as given below. Which page has the highest page rank and thus should be shown first in a list of results that returns these three pages? Explain how you determined this from the data given below.

$\begin{pmatrix} -0.74 \\ -0.55 \\ -0.37 \end{pmatrix}$ with eigenvalue 1.00

$\begin{pmatrix} 0.71 \\ -5.56 \\ 0.71 \end{pmatrix}$ with eigenvalue -0.40

$\begin{pmatrix} 0.71 \\ -5.50 \\ -0.71 \end{pmatrix}$ with eigenvalue -0.50

$|1.00| = 1.00 \longrightarrow$ dominant ∴ has principle eigenvector

$|-0.40| = 0.40$

$|-0.50| = 0.50$

$$\begin{array}{c} K \\ L \\ M \end{array}\begin{pmatrix} -0.74 \\ -0.55 \\ -0.37 \end{pmatrix}$$

$|-0.74| = 0.74 \xrightarrow{\text{largest}}$ $\boxed{\text{K has the highest page rank}}$

$|-0.55| = 0.55$

$|-0.37| = 0.37$