Midterm

Instructions: Please write your answers and show all necessary work on either the provided exam (if you wish to print it), separate sheets of paper, or a tablet. You do not need to print out the exam. You may use your textbook, notes, and resources on CCLE or elsewhere on the internet. You may also use a calculator or computer, including online resources such as CoCalc, Desmos, and WolframAlpha. However, as always, **you must show all of your work to receive full credit** for each problem, and **you must not get help from other people**. All work shown must be your own. If you have a question about the exam at any point during the exam period, you can post it on Campuswire. Questions should only clarify ambiguities and may not reveal your work. (We can't check your work during an exam anyway.)
When you are finished with the exam, take a photo of your answer to each question, and upload them to Gradescope. It is recommended that you start each question on a new page.

1. (10 points) The La Brea tar pits contain fossils of animals that inhabited southern California between 11,000 and 50,000 years ago. These animals include the saber-toothed cat (S), which preyed upon American bison (B) and Western horses (H). The bison and horses, in turn, competed with each other for food. Write a system of differential equations to describe this situation, based on the following assumptions:

   - In the absence of the other species, the horse population will grow logistically with a per-capita growth rate of 0.02 and a carrying capacity of 750. $0.02H\left(1-\frac{H}{750}\right)$ ✓
   - The per-capita birth rate of the bison population will be higher when there are few horses around, and lower when there are many horses to compete with. Use a decreasing sigmoid function for this, with a maximum of 0.5 per year (when there are no horses at all). $0.5\left(\frac{d^n}{d^n+H^n}\right)B$ $H\cdot\cancel{\left(\frac{0.5^n}{0.5^n+B^n}\right)}$
   - An abundance of bison can be harmful to the horse population. When there are many bison, the per-capita death rate of the horses can increase by as much as 0.15 per year. (Use a non-sigmoid saturating function for this, with a maximum of 0.15 per year.) $H\cdot\cancel{\left(\frac{B}{0.15+B}\right)}$ $0.15\cdot\frac{B}{d+B}(H)$
   - Each saber-toothed cat preys on horses at a rate that is a sigmoid function of the horse population, with a maximum of 24 per year. $S\cancel{\left(\frac{H^m}{24^m+H^m}\right)}$ $24\cdot\frac{H^n}{d^n+H^n}(S)$
   - Each saber-toothed cat preys on bison at a rate proportional to the bison population, with a proportionality constant of 0.01. $0.01B\cdot S$
   - The saber-toothed cat population grows logistically, with a per-capita growth rate of 0.015 and a carrying capacity equal to 1/8 of the sum of the horse and bison populations. $0.015S\left(1-\frac{S}{\frac{1}{8}(H+B)}\right)$

     Note: Although several constants are given in this problem, you may want to introduce others in some places. Feel free to do so.

$$H' = 0.02\left(1-\frac{H}{750}\right) - 0.15\left(\frac{B}{d+B}\right)H - 24\frac{H^n}{d^n+H^n}\cdot S$$

$$B' = 0.5\frac{d^n}{d^n+H^n}\cdot B - 0.01B\cdot S$$

$$S' = 0.015S\left(1-\frac{S}{\frac{1}{2}(H+S)}\right)$$

2. (8 points) During the Great Toilet Paper Shortage of 2020, many stores found themselves running out of paper products. Use the assumptions below to write a differential equation model for T, the amount of toilet paper stores have in stock.

- The rate at which shoppers buy toilet paper is a steep declining sigmoid function of the amount on store shelves, with a maximum of 1.
- Stores get toilet paper from warehouses, but it takes time to ship it from the warehouse to the store. Therefore, the rate at which stores receive toilet paper depends on how much they ordered 3 days ago.
- The amount of toilet paper a store orders is a declining sigmoid function of the amount on the shelf with a maximum of 5. The sigmoid is not steep.

$$\left(\frac{5}{5+T(t-3)^n}\right) \longrightarrow \boxed{T} \longrightarrow \left(\frac{1}{1+T^n}\right)$$

$$\boxed{T' = \left(\frac{5}{1\cancel{5}+T(t-3)^{m}}\right) - \left(\frac{1}{1+T^n}\right)}$$

$n =$ steepness of toliet paper bought

$m =$ stepness of amount toliet paper ordered

3. (a) (4 points) You are studying hormone concentrations in a plant and notice that they seem to follow a 24-hour cycle. How could you determine if the oscillations are neutral or limit cycle oscillations?

also include: If you perturb the system, the system will return to oscillation

After creating a time series which measures the hormone concentration of the plant with repect to its time in the 24 hour cycle, you can stimulate the trajectories of the system. If the trajectories appear to be going towards a specific point — it is a neutral oscillation. If trajectories appear to go to a limit cycle, where trajectories inside and outside of the limit cycle go around the cycle until the end up following its pattern (stable limit cycle).

(b) (4 points) After further study, you notice that the oscillations only occur at high temperatures. Furthermore, the hormone you are studying inhibits its own production. In dynamical terms (no biological detail necessary), what aspect of hormone production might the increase in temperature have changed and in what direction (increasing or decreasing)?

Due to the hormone inhibiting its own production, it can be identified as a form of negative feedback. when n (steepness) of the inhibition is increased, stable oscillations occur at the high temperatures.

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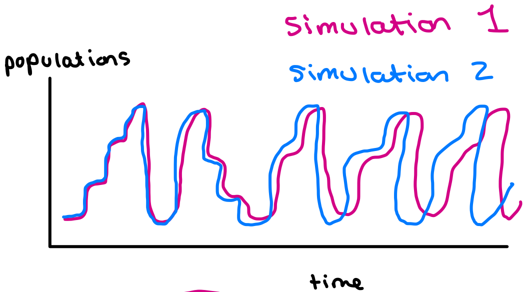
4. (a) (6 points) Suppose all you knew about chaotic systems is that they are deterministic and have attractors. Which other defining property or properties of chaos (boundedness, aperiodicity, and sensitive dependence on initial conditions) could you conclude such a system possessed? Briefly justify your answer.

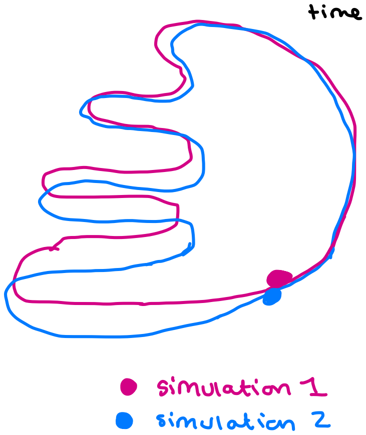
just boundedness

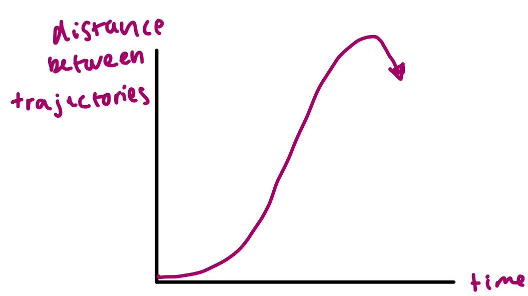
* on next page

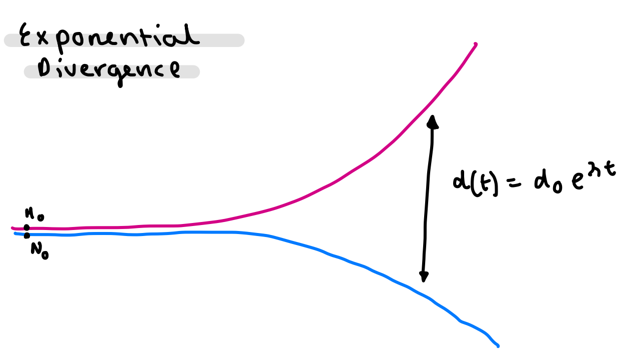
(b) (6 points) You run two simulations of a chaotic system with slightly different initial conditions. Sketch a graph of how the distance between the trajectories would change over the long run. Explain why the graph has the shape that it does.



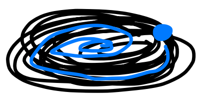
The distance between the trajectories grow over time because of sensitive dependence on initial conditions. sensitive dependence details that the distance between your initial values will grow exponentially with some λ. The initial conditions start out close, but eventually diverge where their behavior does not correspond.







attractor



deterministic

- Due to knowing that chaos has an attractor, we can determine that chaos is <u>bounded</u>, because if we perturb the system the trajectory will always end up in that attractor.

- ~~Knowing that chaos is deterministic, we can say that the system has <u>sensitive dependence on initial conditions</u> because of the definition of determinism; that each state is determined by the previous state. If an initial condition is changed, the future state will be changed.~~

all models have dependence on initial conditions, but they are trivial. Chaotic systems undergo a period of exponential divergence, but it does not go on forever because of boundedness

all deterministic systems have dependence on initial conditions, but most of them do not have exponential sensitive dependence.

2×2

5. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

   (a) (4 points) If $f\left(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$ and $f\left(\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}7\\3\end{smallmatrix}\right]$, what is $f\left(\left[\begin{smallmatrix}0\\10\end{smallmatrix}\right]\right)$? Do this **without** using a matrix.

$$f\left(\begin{bmatrix}2\\5\end{bmatrix}\right) - f\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = f\left(\begin{bmatrix}0\\5\end{bmatrix}\right) \rightarrow$$

$$\begin{bmatrix}7\\3\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\-1\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\10\end{bmatrix}\right) = f\left(2\cdot\begin{bmatrix}0\\5\end{bmatrix}\right) = 2\cdot f\left(\begin{bmatrix}0\\5\end{bmatrix}\right) =$$

$$2\cdot\begin{bmatrix}4\\-1\end{bmatrix} = \boxed{\begin{bmatrix}8\\-2\end{bmatrix}}$$

   (b) (4 points) Write the matrix representing f.

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{f\left(\begin{bmatrix}0\\5\end{bmatrix}\right)}{5} = \begin{bmatrix}4\\-1\end{bmatrix}\cdot\left(\frac{1}{5}\right) = \begin{bmatrix}4/5\\-1/5\end{bmatrix}$$

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{f\left(\begin{bmatrix}2\\0\end{bmatrix}\right)}{2} = \begin{bmatrix}3\\4\end{bmatrix}\cdot\frac{1}{2} = \begin{bmatrix}3/2\\2\end{bmatrix}$$

$$\boxed{\text{Matrix of } f : \begin{bmatrix}3/2 & 4/5\\2 & -1/5\end{bmatrix}}$$

   (c) (4 points) Use the matrix you wrote to find $f\left(\left[\begin{smallmatrix}0\\10\end{smallmatrix}\right]\right)$.

$$\begin{bmatrix}3/2 & 4/5\\2 & -1/5\end{bmatrix}\begin{bmatrix}0\\10\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}8\\-2\end{bmatrix} = \boxed{\begin{bmatrix}8\\-2\end{bmatrix}}$$

6. (6 points) It is tempting to simply multiply matrices term by term. (For example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 21 & 28 \end{pmatrix}$.) Without doing any calculations, explain why such a procedure would not accomplish what we want matrix multiplication to accomplish. *(Hint: Think about functions.)*

Matrix multiplication must follow the procedure of taking a row of matrix A (represented by f) and a column of matrix B (represented by g) multiplying each component then adding the results. This must be the procedure because matrix multiplication takes the form:

$$f \cdot g(x) = f(g(x))$$

We take the vector and apply B to it, then apply A in order to calculate the new matrix.