Midterm 2

Instructions: Please write your answers and show all necessary work on either the provided exam (if you wish to print it), separate sheets of paper, or a tablet computer. You do not need to print out the exam. You may use your textbook, notes, and resources on CCLE or elsewhere on the internet. You may also use a calculator or computer, including online resources such as CoCalc, Desmos, and WolframAlpha. However, as always, **you must show all of your work to receive full credit** for each problem, and **you must not get help from other people**. All work shown must be your own. If you have a question about the exam at any point during the exam period, you can email your question to LS30BS2020@gmail.com. When you are finished with the exam, take a photo of your answer to each question, and upload them to Gradescope. Note that you will upload a separate file for each question! We recommend that you start each question on a new page.

1. (10 points) Great white sharks are vicious man-eating beasts*, the sight of which provokes humans to run away (despite the fact that they're in the water and therefore can't run). Research data has yielded the following information:

   - The shark population can be subdivided into baby sharks (B), mommy sharks (M), and grandma sharks (G).
   - Baby sharks take approximately 22 years to reach reproductive maturity, so each year about 4% of them mature into mommy sharks.
   - Each year, about 5% of mommy sharks mature into grandmas.
   - Each year, approximately 9% of baby sharks die, and 3% of mommy sharks die.
   - On average, a mommy shark will give birth to 2.4 baby sharks each year.
   - Grandma sharks are usually too old to reproduce, but do so occasionally. So on average, each grandma shark gives birth to 0.15 baby sharks each year.
   - 93% of grandma sharks survive from one year to the next.

   Set up a discrete-time model for the shark population, based on the assumptions above. Write your answer in the form [next state] = M[current state].

*Actually, there are only 4 deaths caused by shark attacks worldwide per year.

Question 1 continues on the next page...

2. Nitrogen is a necessary element for life but is scarce in the inner solar system outside Earth. To design ways to manage nitrogen in a potential space settlement, we can start with a basic model that contains plants (P) grown for food, humans (H), and decomposers (D) that break down plant and human waste. The model keeps track of how much nitrogen each part of the system contains. It is based on the following assumptions:

   - Each month, 20% of decomposer nitrogen goes to plants.
   - Since the human population is constant, humans consume a constant 10% of the plant nitrogen in a month.
   - 15% of the nitrogen in plants is in inedible parts, and thus goes to decomposers.
   - The amount of nitrogen humans excrete in urine is proportional to the amount in their bodies, with a proportionality constant of 0.01. This nitrogen goes to decomposers.

   (a) (8 points) Using these assumptions, write a *differential equation* model of the system.
   (Hint: Draw a flow diagram.)

Question 2 continues on the next page...

(b) (2 points) Is the model you just wrote linear? Why or why not?

3. A computer is made up of parts, such as the case, CPU chip, memory chips, etc.

   (a) (5 points) Is the total cost of manufacturing a computer (including overhead costs, wages, etc.) a linear function of the costs of the parts? Justify your answer.

   (b) (5 points) Is the weight of the computer a linear function of the weights of the parts? Justify your answer.

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4. (a) (4 points) Suppose $f: \mathbb{R}^2 \to \mathbb{R}^3$ is linear. If

$$f(\begin{bmatrix} -2 \\ 0 \end{bmatrix}) = \begin{bmatrix} -8 \\ 0 \\ 6 \end{bmatrix} \text{ and } f(\begin{bmatrix} 1 \\ 3 \end{bmatrix}) = \begin{bmatrix} 7 \\ 3 \\ -9 \end{bmatrix},$$

find the matrix representing f.

(b) (2 points) Calculate $f(\begin{bmatrix} 2 \\ -2 \end{bmatrix})$ using the same function as above.

Question 4 continues on the next page...

(c) (6 points) Suppose $g: \mathbb{R}^3 \to \mathbb{R}^2$ is also linear. If

$$g\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \qquad g\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix}, \qquad \text{and} \qquad g\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\5\end{bmatrix},$$

what is $f \circ g$? What about $g \circ f$? If one or both are not possible, explain why. If possible, find the matrix that represents the composition.

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5. Suppose M is a 3×3 matrix, and you know the following:

- $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of M with eigenvalue 3.
- $\vec{w} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is also an eigenvector of M with eigenvalue 3.

(a) (6 points) Compute $M \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$.

(Hint: First compute $M\vec{v}$ and $M\vec{w}$.)

(b) (4 points) Is $\begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$ an eigenvector of M? If so, what is the corresponding eigenvalue?

Bonus: (2 points) If you're given two eigenvectors corresponding to the same eigenvalue, is their sum ever/always an eigenvector? Justify your answer.

6. Consider the discrete-time model

$$\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = \begin{bmatrix} -9X_t + 5Y_t \\ -15X_t + 11Y_t \end{bmatrix}.$$

(a) (3 points) Find the eigenvalues of the matrix M, where M is the matrix that represents this model. Make sure to show your work!

(b) (7 points) For each of the eigenvalues you found, find the equation of the "eigenline" *and* a sample eigenvector. Show your work!

7. (8 points) A rancher wants to purchase some number of cows and some number of horses as an investment. She is considering purchasing up to 15 cows (which cost \$2000 each) and up to 10 horses (which cost \$5000 each). She decides to create a table showing the *total cost* of buying a specific number of cows and horses, like the following:

	Horses:			
Cows:	0	1	2	$\cdots$
0	\$0	\$5000	\$10000	$\cdots$
1	\$2000	\$7000	\$12000	$\cdots$
2	\$4000	\$9000	\$14000	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

After the first few entries, this gets tedious, but as a life science major, she learned that she can do this programmatically.

Below is a jumbled collection of some lines of code. Put these lines of code in the *correct order* and with the *correct indentation* to print a matrix of values as in the table above.

Lines of code:

A. `for j in srange(16):`

B. `return output`

C. `for i in srange(11):`

D. `output = 2000*a + 5000*b`

E. `cost_mat = zero_matrix(RR, 16, 11)`

F. `print(cost_mat)`

G. `cost_mat[j, i] = total_cost(j, i)`

H. `def total_cost(a, b):`

Note that you don't have to write out the code, just the letters of the lines. For example, you could answer something like the following:

Question 7 continues on the next page...

Question 7 continued. . .

A
B
C
D
E
F
G
H