LS 30B-3 Final Exam

Elizabeth Frances Elton

TOTAL POINTS

87.5 / 96

QUESTION 1

1 Flu Model 6.5 / 8

✓ **- 0.5 pts** Assumption 1: Missing entire term from either S' or I' equation; included sigmoid function, or missed variables, or included extra variables in both equations

✓ **- 1 pts** Assumption 7: Missing 0.005M in either S' or M' equation

QUESTION 2

Oscillations & Attractors 15 pts

2.1 Center 2 / 3

✓ **- 0.5 pts** Partially incorrect oscillation type, including "stable" (correct: neutral or center; "stable" =/= "neutrally stable", see page 176 in book)

✓ **- 0.5 pts** Partially incorrect attractor type, including "center" (correct: none)

2.2 Stable Spiral 3 / 3

✓ **- 0 pts** Correct

2.3 Unstable Spiral 2.5 / 3

✓ **- 0.5 pts** Partially incorrect or incomplete attractor type, including "unstable spiral" or "unstable EP" only (correct: no attractor)

2.4 Limit Cycle Attractor 3 / 3

✓ **- 0 pts** Correct

2.5 Chaos 3 / 3

✓ **- 0 pts** Correct

QUESTION 3

Composition of Linear Functions 9 pts

3.1 Make Matrix for f 3 / 3

✓ **- 0 pts** Correct

3.2 fog 3 / 3

✓ **- 0 pts** Correct

3.3 gof 3 / 3

✓ **- 0 pts** Correct

QUESTION 4

Linear Functions and Eigen-Stuff 12 pts

4.1 Could f be linear? 3 / 3

✓ **- 0 pts** Correct

4.2 Matrix for 7 state variables 3 / 3

✓ **- 0 pts** Correct

4.3 Mathematical equation for eigenvalues & eigenvectors 2 / 2

✓ **- 0 pts** Correct

4.4 Eigenvectors of g 3 / 4

✓ **- 1 pts** Incorrect or missing eigenvalue (L=3)

QUESTION 5

Gorilla Matrix Model 16 pts

5.1 Writing Model 8 / 8

✓ **- 0 pts** Correct

5.2 Modify for Ebola 2 / 2

✓ **- 0 pts** Correct

5.3 Sequence of Years 2 / 2

✓ **- 0 pts** Correct (M^4*E*M^2)

5.4 Decline? 1 / 4

✓ **- 1 pts** Incorrect or no usage of eigenvalue; OR incorrect usage of eigenvector instead of eigenvalue

✓ **- 1 pts** No mention of population declining / going extinct

✓ **- 1 pts** Incorrect or no mention of decline by 5.6% (or 0.056) per year

QUESTION 6

Dynamics of Mouth Bacteria 12 pts

6.1 Find Trivial Equilibrium Points 3 / 3

✓ - 0 pts Correct - (0,0), (0,15), (12,0)

6.2 Find Jacobian in Terms of M and S 4 / 4

✓ - 0 pts Correct

6.3 Classify Non-Trivial Equilibrium Point & Coexist 4.5 / 5

✓ - 0.5 pts c) Correct characteristic equation - incorrect calculation of eigen values

QUESTION 7

7 Linear Approximation for BMI 8 / 9

✓ - 1 pts Wrong delta-W or delta-H (or don't use)

QUESTION 8

Hopf Bifurcation 7 pts

8.1 Change in Equilibrium Points 4 / 4

✓ - 0 pts Correct

8.2 Parameter Value 3 / 3

✓ - 0 pts Correct

QUESTION 9

9 Optimization of Drug Dose 8 / 8

✓ - 0 pts Correct

Final Exam

First Name: Elizabeth

Last Name: Ellon

Last 6 digits of UID: 986788

Section # (TA): Ming 3C

By signing below, you confirm that you did not cheat on this exam. No exam without a signature will be graded.

Signature:

Instructions: Do not open this exam until instructed to do so. You will have 2 hours and 50 minutes to complete the exam. Please print the last 6 digits of your student ID number above and on each page of the exam. You may not use books, notes, or any other material to help you. You may use a scientific or 4-function calculator for this exam. Please make sure your phone is silenced and stowed at the front of the room. Please write only on the space below each problem. We are providing plenty of space for each problem, so you should not need additional space or tiny handwriting to answer any of the problems. However, you can use pages 15 and 16 or request additional paper for scratch paper which will not be graded. After you start the exam, tear off pages 15 and 16 and turn them in separately.

Please do not write below this line.

Problem	Max	Score
1	8	
2	15	
3	9	
4	12	
5	16	
6	12	
7	9	
8	7	
9	8	
Total	96	

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1. (8 points) It's flu season, so let's model the number of susceptible (S), infected (I) or immune (M) people on a college campus. Use the assumptions to write a differential equation model of flu dynamics on a college campus. Feel free to make up parameters if necessary. If you forget the formula for a function, you can sketch its graph for partial credit. All rates are per day.
   - If a susceptible person meets an infected one, the susceptible person becomes infected with probability 0.02. (In other words, susceptible people become infected at a per-capita rate that is proportional to the number of infected people with proportionality constant 0.02.)
   - Infected people die at a per-capita rate of 1/1000.
   - Infected people recover at a per-capita rate of 0.1.
   - When people recover from the flu, they become immune to it.
   - Susceptible people can become immune by being vaccinated. Because people are more likely to get vaccinated if they see a lot of others getting sick, the per-capita vaccination rate is a steep sigmoid function of the infected population that saturates at a maximum of 0.75.
   - The flu vaccine takes 14 days to become fully effective, so a vaccinated person is not immune until 14 days after vaccination.
   - Immune people lose immunity at a per-capita rate of 0.005.

$I \to M$

$S \to M$

$\frac{x^n}{1+x^n}$ $n \gg 1$

susceptible $\to$ immune

large 'n' = steep function

$$S' = 0.02SI - \frac{0.75I^{10}}{1+I^{10}}S(t-14)$$

$$I' = -\frac{1}{1000}I - 0.1I$$

$$M' = 0.1I + \frac{0.75I^{10}}{1+I^{10}}S(t-14) - 0.005M$$

$\downarrow$

$$\boxed{\begin{aligned} S' &= 0.02SI - \frac{0.75I^{10}}{1+I^{10}} \cdot S(t-14) \\ I' &= -0.101I \\ M' &= 0.1I + \frac{0.75I^{10}}{1+I^{10}} \cdot S(t-14) - 0.005M \end{aligned}}$$

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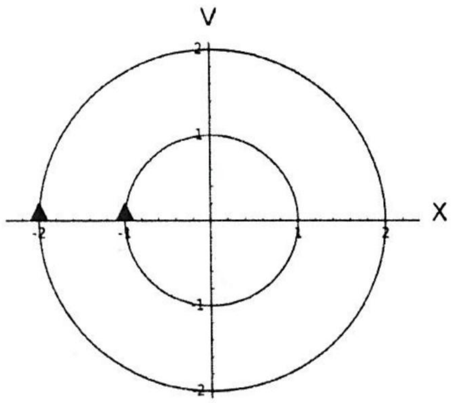
2. (15 points) For each of the model trajectories shown below:
    a. Identify the type of oscillations (if any).
    b. Identify the type of attractor (if any).
    c. Draw an approximate time series graph for one of the variables.

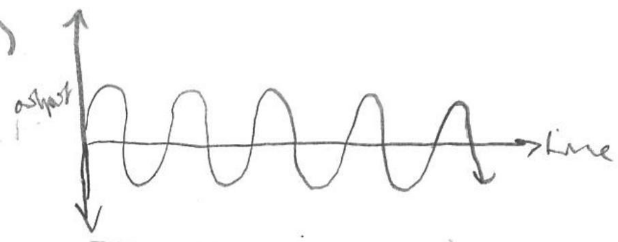
i.



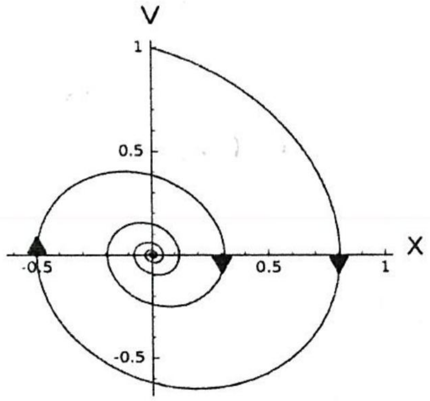
a.) persistent oscillations

b.) center

c.)



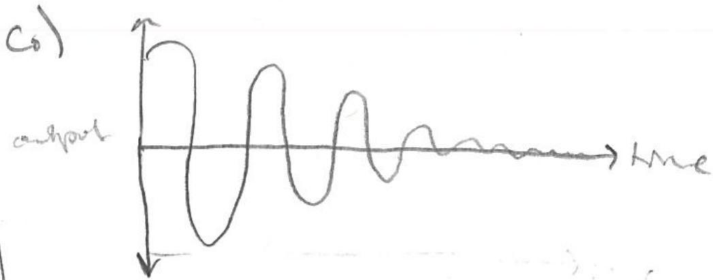
ii.



a.) damped oscillations

b.) stable spiral

c.)



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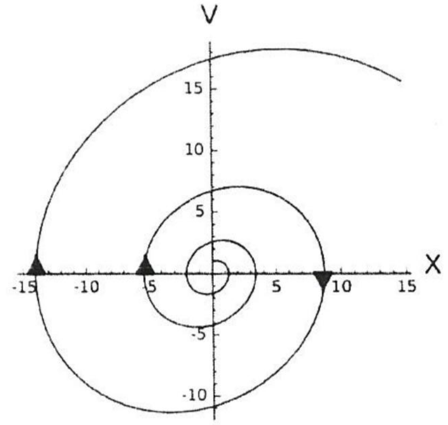
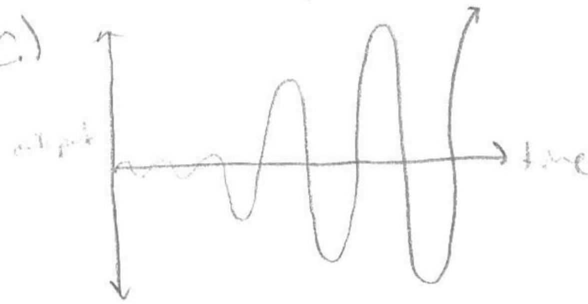
iii.



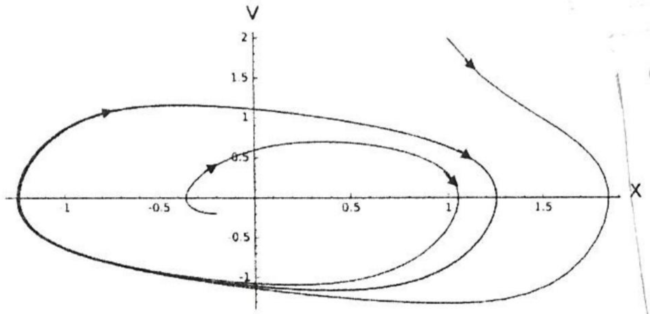
a.) increasing oscillations

b.) unstable spiral

c.)



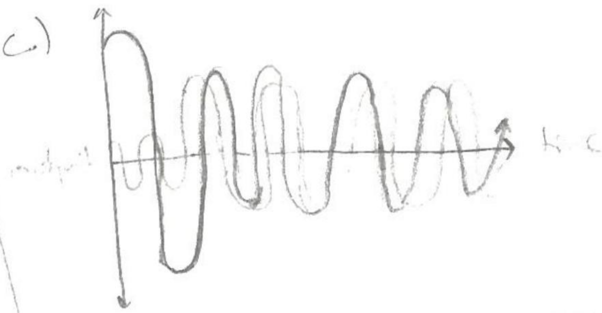
iv.



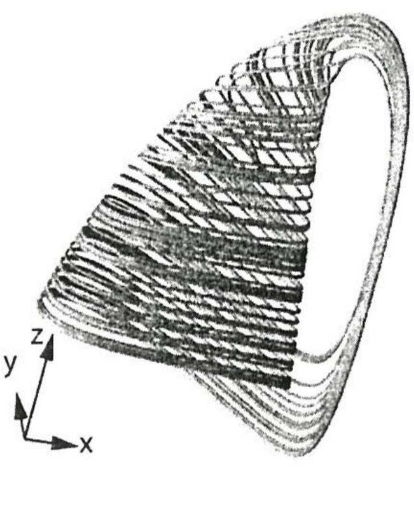
a.) stable oscillations

b.) limit cycle attractor

c.)



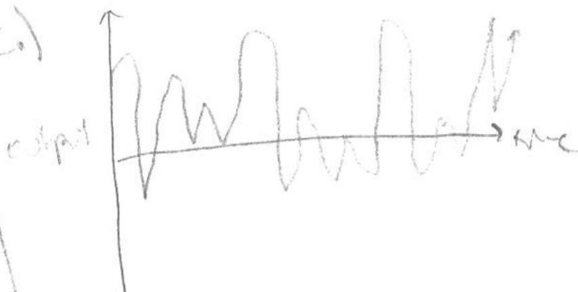
v.



a.) chaotic behavior (not oscillato

b.) strange attractor

c.)



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3. We have two linear functions, $f: \mathbb{R}^2 \to \mathbb{R}^2$ and $g: \mathbb{R}^3 \to \mathbb{R}^2$. The matrix representing g is

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}.$$

a. (3 points) $f\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ and $f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$. What is the matrix representing f?

$f\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = 3f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \therefore f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\boxed{f = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}}$

b. (3 points) Find the matrix of $f \circ g$, or explain in terms of functions (not matrices) why it does not exist. Note: if you were unable to find the matrix representing f in (a), make up an arbitrary matrix that represents a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ and either calculate $f \circ g$ or explain why it does not exist.

$f \circ g = f(g(x)) = \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^2 \to$ can exist

$\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 3-2 & 6-4 & 9-6 \\ 5-2 & 10-4 & 15-6 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}}$

c. (3 points) Find the matrix of $g \circ f$, or explain in terms of functions (not matrices) why it does not exist. Note: if you were unable to find the matrix representing f in (a), make up an arbitrary matrix that represents a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ and either calculate $g \circ f$ or explain why it does not exist.

$g \circ f = g(f(x)) =$

$\underbrace{(\mathbb{R}^2 \longrightarrow \mathbb{R}^2)}_{f} \cancel{\ } \underbrace{(\mathbb{R}^3 \longrightarrow \mathbb{R}^2)}_{g}$

– the output of f is $\mathbb{R}^2$, which g cannot take in as an input. Therefore $g \circ f$ cannot exist

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4. The parts of this problem are independent.
   a. (3 points) Suppose that $f: \mathbb{R}^2 \to \mathbb{R}^3$ is a function for which

$$f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}3\\1\\2\end{pmatrix} \text{ and } f\left(\begin{pmatrix}3\\6\end{pmatrix}\right) = \begin{pmatrix}7\\3\\6\end{pmatrix}$$

   Could f be a linear function? Why or why not?

Linear definition: $f(a(x)) = af(x)$

$\therefore 3f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = f\left(3\begin{pmatrix}1\\2\end{pmatrix}\right) = f\left(\begin{pmatrix}3\\6\end{pmatrix}\right)$

$3f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = 3\begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}9\\3\\6\end{pmatrix} \neq \begin{pmatrix}7\\3\\6\end{pmatrix}$

$\therefore$ Function doesn't follow the definition and therefore could not be linear.

   b. (5 points) Given a matrix M representing a system of linear differential equations with 7 state variables,
      i. How many rows and columns does the matrix M have?

7 rows and 7 columns

      ii. How many eigenvalues does the matrix M have?

7

      iii. How many elements does each eigenvector have?

7 $\longrightarrow$ i.e. $\begin{pmatrix}a\\b\\c\\d\\e\\f\\g\end{pmatrix}$

      iv. Give a mathematical equation involving M for the relationship between each eigenvector and its corresponding eigenvalue.

$M\vec{v} = \lambda\vec{v}$ — eigenvector

$\uparrow$ eigenvalue

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c. (4 points) Suppose $g: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function for which

$$g\left(\begin{pmatrix}3\\5\end{pmatrix}\right) = \begin{pmatrix}9\\15\end{pmatrix}$$

$$g\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}5\\7\end{pmatrix}$$

$$g\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix}$$

$$g\left(\begin{pmatrix}1\\3\end{pmatrix}\right) = \begin{pmatrix}-1\\1\end{pmatrix}$$

Which of the above input vectors are eigenvectors of the matrix representing g? What are their corresponding eigenvalues? (Hint: you should not need to do any complicated calculations.)

$\boxed{\begin{pmatrix}3\\5\end{pmatrix}}$ and $\boxed{\begin{pmatrix}1\\2\end{pmatrix}}$ are eigenvectors of g.

The eigenvalue of $\begin{pmatrix}3\\5\end{pmatrix}$ is $\boxed{\frac{5}{3} \text{ or } 1.67}$.

The eigenvalue of $\begin{pmatrix}1\\2\end{pmatrix}$ is $\boxed{2}$.

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5. Gorillas can be classified as immature (I), mature (M), or senior (S).
   a. (8 points) Write a discrete-time matrix model of a gorilla population in a typical year using the following assumptions.
      - About 7% of mature gorillas will give birth to one offspring each year.
      - Rarely, a senior gorilla will give birth. Assume that on average only 0.5% of seniors produce offspring each year.
      - A gorilla reaches maturity at 12 years of age, so about 8% of immature gorillas reach maturity each year.
      - A gorilla is considered a senior when it reaches about 28 years, or about 16 years after reaching maturity. Thus, about 6% of mature gorillas become seniors each year.
      - The per-capita death rate of mature gorillas is 4%. It is higher for immature gorillas and seniors: approximately 12% of immature gorillas die each year, and 9% of seniors die.

      Your final answer should look something like

      (next state)= M(current state)

      (where M is a matrix and appropriate mathematical expressions in place of the words).

$$\Delta I = .07M + .005S - .08I - .12I$$
$$\Delta M = .08I - .06M - .04M$$
$$\Delta S = .06M - .09S$$

$$I_{n+1} = .8I_n + .07M_n + .005S_n$$
$$M_{n+1} = .08I_n + .9M_n$$
$$S_{n+1} = .06M_n + .91S_n$$

$$\boxed{\begin{pmatrix} I_{n+1} \\ M_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{bmatrix} 0.8 & 0.07 & 0.005 \\ 0.08 & 0.9 & 0 \\ 0 & 0.06 & 0.91 \end{bmatrix} \begin{pmatrix} I_n \\ M_n \\ S_n \end{pmatrix}}$$

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b. (2 points) Like humans, gorillas of all ages experience high fatality rates during outbreaks of Ebola. Modify your matrix model to reflect a year with an Ebola outbreak affecting gorillas. Call this matrix E.

$$\boxed{E = \begin{bmatrix} 0.3 & 0.07 & 0.005 \\ 0.08 & 0.4 & 0 \\ 0 & 0.06 & 0.41 \end{bmatrix}}$$

(each stage experiences extra 50% fatality rate)

c. (2 points) Write an expression for the matrix that would represent a sequence of 2 typical years followed by a year with an Ebola outbreak followed by 4 typical years. You can represent the matrix for a typical year as M and the matrix for a year with an Ebola outbreak as E. You do not need to do any calculations for this subproblem.

$$Q = \boxed{M^4 E M^2}$$

d. (4 points) Suppose the matrix for a typical year in part (a) had the following (approximate) eigenvectors and corresponding eigenvalues:

$$\begin{pmatrix} 1.000 \\ 1.823 \\ 3.233 \end{pmatrix} \text{ with eigenvalue } 0.944$$

$$\begin{pmatrix} 1.000 \\ 15.238 \\ -192.308 \end{pmatrix} \text{ with eigenvalue } 0.905$$

$$\begin{pmatrix} 1.000 \\ -0.575 \\ 0.231 \end{pmatrix} \text{ with eigenvalue } 0.761$$

Do you expect the population to persist or go extinct in the long run? How much is the population growing or declining each year? Explain your answers using the information in this problem.

The dominant eigenvalue 0.944 has corresponding eigenvector $\begin{pmatrix} 1 \\ 1.823 \\ 3.233 \end{pmatrix}$ which tells us this population will persist with no growth/decay per year for immature gorillas, 82.3% growth per year for mature gorillas, and 223.3% growth per year for senior gorillas.

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6. Many bacteria live in the biofilm (plaque) between and on our teeth and compete for resources in various ways. Two of the most important species are *Streptococcus mutans* (M, causes tooth decay) and *Streptococcus sanguinis* (S, inhibits tooth decay). Suppose their interaction in a patient's mouth is described by the following model:

$$M' = 24M - 2M^2 - 3MS$$
$$S' = 15S - S^2 - 3MS$$

a. (3 points) This model has only one non-trivial equilibrium point, $(M^*, S^*) = (3,6)$. Find the coordinates of the other three equilibrium points of this system.

$0 = M(24 - 2M - 3S)$ $\qquad$ $0 = S(15 - S - 3M)$

$0 = 24 - 2M - 3(0)$ $\qquad$ $0 = 15 - S - 3(0)$

$2M = 24$ $\qquad$ $0 = 15 - S$

$M = 12$ $\qquad$ $S = 15$

$(0,0)$
$(0,15)$
$(12,0)$

b. (4 points) Find the Jacobian matrix of this model in terms of M and S.

$$J = \begin{bmatrix} \frac{dM'}{dM} & \frac{dM'}{dS} \\ \frac{dS'}{dM} & \frac{dS'}{dS} \end{bmatrix} = \begin{bmatrix} 24 - 4M - 3S & -3M \\ -3S & 15 - 2S - 3M \end{bmatrix}$$

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c. (3 points) Classify the type of the non-trivial equilibrium point, $(M^*, S^*) = (3,6)$. Show your work.

$$J_{(3,6)} = \begin{bmatrix} 24-4M-3S & -3M \\ -3S & 15-2S-3M \end{bmatrix}_{(3,6)} =$$

$$= \begin{bmatrix} 24-4(3)-3(6) & -3(3) \\ -3(6) & 15-2(6)-3(3) \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -18 & -6 \end{bmatrix}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{0 \pm \sqrt{-4(36-162)}}{2} = \frac{\pm\sqrt{504}}{2}$$

$$= \frac{\pm 22.4499}{2} \quad \lambda_1 = 11.22 \quad \lambda_2 = -11.22$$

one positive & one negative = $\boxed{\text{saddle point}}$

d. (2 points) Can the two bacteria populations coexist in this patient's mouth in the long term? Explain your response using your results from the earlier parts of this problem. You may draw a rough sketch of the phase portrait (which is state space with equilibria and representative trajectories).

Since the non-trivial equilibrium point is a saddle point and the other equilibrium points have either bacteria M or bacteria S or both at 0 population, these two bacteria cannot coexist in this patients mouth

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7. (9 points) We calculate the body mass index (BMI) using weight (W, in pounds) and height (H, in inches) using the formula below:

$$b(W,H) = 703WH^{-2}.$$

A pre-teen patient came to you a month ago when he was 60 inches (5 feet tall) and weighed 150 lbs. After joining the cross-country team, the patient has gained 1 inch in height and lost 5 lbs. Use a linear approximation to calculate the patient's change in BMI. Hint: recall that the linear approximation to a 2D surface at a point is the tangent plane at that point.

$\frac{db}{dW} = 703H^{-2} \qquad \frac{db}{dH} = -1406WH^{-3}$

$\Delta b(W,H) = \frac{db}{dW}\Delta W + \frac{db}{dH}\Delta H$

$\Delta b(W,H) = (703(60)^{-2})(61-60) + (-1406(150)(60^{-3}))(145-150)$

$\Delta BMI = (0.195277)(1) + (-0.976388)(-5)$

$\Delta BMI = 0.195277 + 4.881944$

$\boxed{\Delta BMI = 5.08}$ (3 sig figs)

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8. You are modeling one of the chemical reactions in glycolysis with a differential equation system that has two state variables and only one equilibrium point. For high values of the parameter r, the model shows oscillatory behavior and responds to small perturbations with a return to the same oscillations (i.e., stable oscillations).
   a. (4 points) As you change the parameter r in the model, a Hopf bifurcation occurs. How does the equilibrium point change? Describe the equilibrium point before and after the bifurcation both verbally and in terms of eigenvalues of the Jacobian matrix representing the system.

The equilibrium point goes from unstable spiral to stable spiral (passing through center equilibrium

This means that the eigenvalues of the Jacobian go from having positive real parts to negative real parts, but always being complex numbers.
The Hopf bifurcation will occur exactly when the real part of the eigenvalues equal zero, because it's passing from positive to negat

   b. (3 points) You are able to solve for the equilibrium point as a function of r, and you find that the Jacobian matrix of the system at that equilibrium point is (in terms of the parameter r):

$$J = \begin{bmatrix} r & 6r \\ -3 & r-6 \end{bmatrix}$$

At exactly what value of r does the Hopf bifurcation occur? Make sure you show how you found that value. Note that r is a reaction constant and must be greater than 0.5.

The Hopf bifurcation occurs as the real part of the complex eigenvalues goes from positive to negative re, when it equals zero. Therefore this can be calculated by saying

$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Hopf bifur. when real part of characteristic equation $a + d = 0$

$\therefore r + r - 6 = 0$

$2r = 6$

$\boxed{r = 3}$ when Hopf bifur. occurs.

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9. (8 points) Doctors often use optimization to determine the best doses of multiple drugs to treat a patient's disease. For example, combining multiple drugs such as interferon-α and ribavirin is a standard therapy for enhancing the antiviral effects and reducing the risk of drug-resistant Hepatitis C virus. Suppose that the virus replication rate is given by:

$$f(I, R) = 2I^2 - 16I + 3R^2 - 18R + 15$$

Find the doses of interferon-α (I) and ribavirin (R) that will minimize the virus replication rate. For full credit, show how you found the levels and show that this critical point really is a minimum.

$\frac{\partial f}{\partial I} = 4I - 16$ $\quad \frac{\partial f}{\partial R} = 6R - 18$

$0 = 4I - 16$ $\quad 0 = 6R - 18$

$16 = 4I$ $\quad 18 = 6R$

$\underline{4 = I}$ $\quad \underline{3 = R}$ $\quad \rightarrow$ point (I, R) $(4, 3)$

$$H = \begin{bmatrix} \frac{d^2 f}{dI^2} & \frac{d^2 f}{dR\,dI} \\ \frac{d^2 f}{dI\,dR} & \frac{d^2 f}{dR^2} \end{bmatrix}_{(4,3)} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}_{(4,3)} \quad \lambda_1 = 4,\ \lambda_2 = 6 \quad > \text{minimum}$$

- Both eigenvalues of the Hessian matrix at the critical point (4, 3) are positive, so the critical point is really a minimum. Therefore, a dose of 4 I and 3 R will minimize the virus replication rate.