LS 30A: MATHEMATICS FOR LIFE SCIENTISTS
FALL 2017 - LECTURE 1
Jukka Keranen

MIDTERM SOLUTIONS

Your Name

Your Student ID number

Your TA Section

By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

____

INSTRUCTIONS

• Please do **not** open this booklet until you are told to do so.

• In addition to basic writing instruments, you are allowed to use a non-programmable calculator.

• Your cell phone must be **turned off completely** and stowed away where you cannot see it.

• No books or notes.

• If you have a question at any time during the exam, please raise your hand.

• You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.

• Make sure to write legibly. **Illegible work will not be graded.**

• Make sure to show all your work and justify your answers fully.

SCORE

1. ______

2. ______

3. ______

4. ______

5. ______

6. ______

TOTAL ______

1. (10 pts) Raccoons inhabit the UCLA campus. Write down a differential equation for the number R of raccoons living at the UCLA campus, using the following assumptions.

1. Every year, m raccoons come from the Santa Monica Mountains and join the UCLA population.
2. The raccoon per capita birth rate is proportional to the number of trash cans, c (their primary food source), with a constant of proportionality b.
3. The raccoon per capita death rate is proportional to the ratio of the number of raccoons to the number of trash cans, with a constant of proportionality d.
4. Every year, s raccoons emigrate from the UCLA campus to the USC campus.

Solution

$$R' = m + bcR - d\left(\frac{R}{c}\right)R - s$$

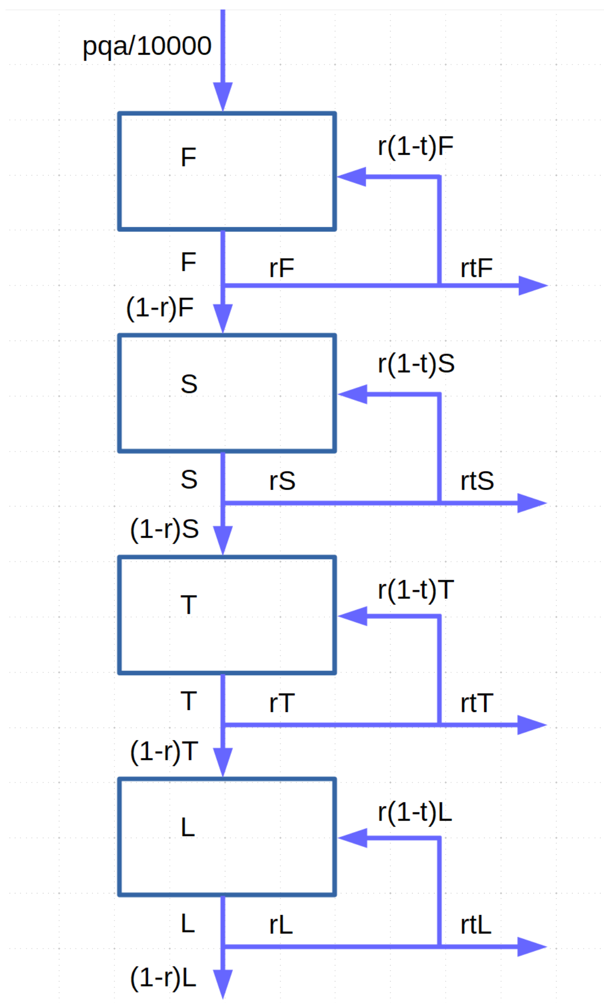
$m, s :$ 1 pt each; $bcR :$ 3 pts, $d\frac{R}{c}R :$ 5 pts

2. *a*) (3 pts) Draw a flow chart for the UCLA student population, using the following assumptions.

Introduce further parameters as necessary.

1. Every student is either a 1^{st}-year student (F), a 2^{nd}-year student (S), a 3^{rd}-year student (T), or a 4^{th}-year student (L).
2. Every year, UCLA receives a certain number applications for admission; call this number a.
3. The UCLA admission rate is p percent.
4. Of the admitted applicants, q percent choose to attend UCLA.
5. Every student starts out as a 1^{st}-year student.
6. A student can graduate only as a 4^{th}-year student.
7. Every year, a certain fraction of students in each year level fail to advance to the next year level; to advance from the 4^{th} year level means that that you graduate. You may assume that it is the same fraction for all year levels.
8. Of those who fail to advance, a certain fraction will leave UCLA without a degree, while the rest will remain at their current level for another year. You may assume that it is the same fraction for all year levels.

Solution



Representing the fractions correctly: 2 pts; the rest, 1 pt.

b) (5 pts) Write down a model for the UCLA student population, using the assumptions on the previous page.

For each parameter of your model, indicate the range of its possible values.

Solution The change equations:

$F' = \left(\frac{p}{100}\right)\left(\frac{q}{100}\right) a + r(1-t)F - F$

$S' = (1-r)F + r(1-t)S - S$

$T' = (1-r)S + r(1-t)T - T$

$L' = (1-r)T + r(1-t)L - L,$

$r =$ the fraction who fail to advance; $t =$ the fraction who leave without a degree

$0 \leqslant p, q \leqslant 100$ (in $\mathbb{Q}$); $a \geqslant 0$ (in $\mathbb{Z}$); $0 \leqslant r, t \leqslant 1$ (in $\mathbb{Q}$)

Each change equation: 1 pt, the parameter ranges (even without the specifications of the sets): 1 pt

c) (2 pts) By using the notation from the first two parts of this problem, write down an equation representing the condition that **the overall student population is neither growing nor declining**.

Solution The inflow into the overall population must equal the net outflow from the overall population, so

$$\left(\frac{p}{100}\right)\left(\frac{q}{100}\right) a = rtF + rtS + rtT + rtL + (1-r)L$$

Each side of the equation: 1 pt

No simplification is required.

In the absence of an equation, 1 pt may be given for the correct basic idea, expressed verbally.

3. (10 pts) In a certain town, there lives a population of N nerds. The nerds patronize comic book stores, of which there are C in this town. The nerds are harassed by mean hipsters, of which there are H. Write down a model for the nerd-hipster-comic book store system, using the following assumptions.

- Since comic book stores rely on nerds for business, the rate at which new comic book stores are opened is proportional to the number of nerds, with a proportionality constant of 0.001.
- Pairs of hipsters amuse themselves by making fun of comic book stores, negatively affecting the image of the stores. So, comic book stores go out of business at a per store rate proportional to the likelihood of two hipsters running into each other in front of a store, with a proportionality constant of 0.002.
- Nerds enjoy living in a town with many other nerds and many comic book stores, so nerds move into town at a rate proportional to the number of nerds times the number of comic book stores, with a proportionality constant of 0.03.
- Nerds move out of town due to being hired by Google at a per capita rate of 0.04 per year.
- In addition, each year, nerds move out of town in order to get away from the hipsters at a per capita rate proportional to the number of hipsters, with a proportionality constant 0.05.
- Hipsters like to live in a town with few comic book stores, so hipsters move into town at a rate proportional to the reciprocal of the number of comic book stores, with a proportionality constant of 0.06.
- Hipster dislike living in a town with many nerds, so hipsters move out of town at a per capita rate proportional to the number of nerds, with a proportionality constant of 0.07.
- Every once in a while, a hipster walks into a comic book store by chance, likes what they find in there, and becomes a nerd. The probability of a hipster liking what they find in a comic book store is 0.0008.

Solution

$C' = 0.001N - 0.002H^2C^2$ (3 pts)

$N' = 0.03NC + 0.0008HC - 0.04N - 0.05HN$ (4 pts)

$H' = 0.06C^{-1} - 0.07NH - 0.0008HC$ (3 pts)

4. *a*) (2 pts) By using the concepts we have learned in this course, state the definition of a *vector field*.

Solution Let M be a model. A vector field is a function from the state space of M into the tangent space of M, given by the change equations of the model: if $X_1' = f_1(X_1, ..., X_n), ..., X_n' = f_n(X_1, ..., X_n)$ are the change equations, then the vector field of M is given by

$$f(X_1, ..., X_n) = (f_1(X_1, ..., X_n), ..., f_n(X_1, ..., X_n)).$$

The phrase (or any reasonable permutation of) "function from the state space into the tangent space": 1 pt

The concrete description in terms of the change vectors: 1 pt

b) (8 pts) Let R be Romeo's love for Juliet (or hate if negative), and let J be Juliet's love for her Romeo (or hate if negative). Assume that the Romeo-Juliet system is modeled by

$$\begin{cases} J' = 0.2J + 0.4R \\ R' = 0.5R - 0.1J^2 \end{cases}$$

Compute eight vectors from the vector field of this model with $-4 \leqslant J \leqslant 4$ and $-4 \leqslant R \leqslant 4$. Then draw those vectors in the diagram provided on the next page. Spread them out so as to give a reasonable representation of the field.

Solution One possible choice of vectors would be

$R'(2, 0) = 1$; $J'(2, 0) = 0.8$

$R'(2, 2) = 0.6$; $J'(2, 2) = 1.2$

$R'(0, 2) = -0.4$; $J'(0, 2) = 0.4$

$R'(-2, 2) = -1.4$; $J'(-2, 2) = -0.4$

$R'(-2, 0) = -1$; $J'(-2, 0) = -0.8$

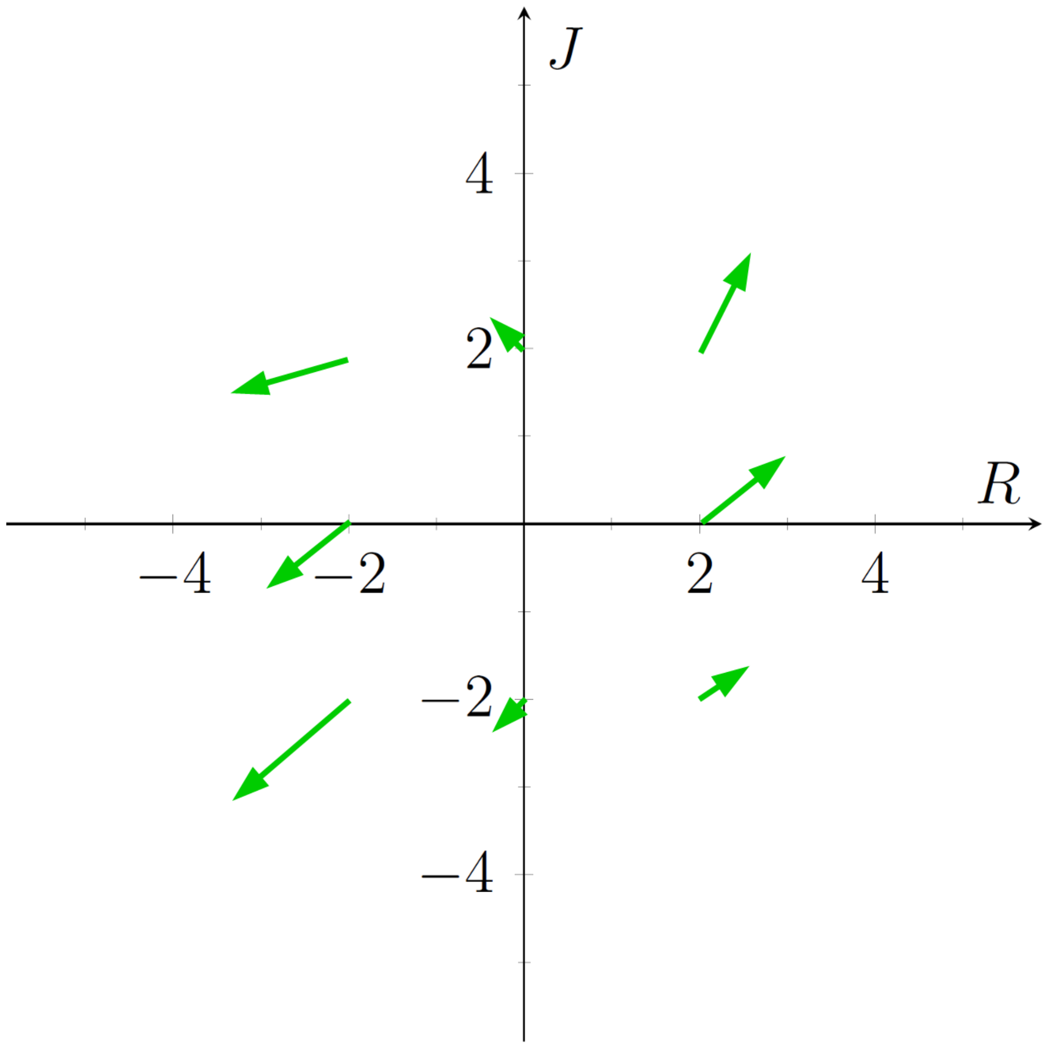
$R'(-2, -2) = -1.4$; $J'(-2, -2) = -1.2$

$R'(0, -2) = -0.4$; $J'(0, -2) = -0.4$

$R'(2, -2) = 0.6$; $J'(2, -2) = 0.4$

The components of the vector along with the drawing: 1 pt

Draw your vectors here:



5. *a*) (2 pts) State the formula for Euler's method in the case the number n of state variables is 2.

Solution Let X and Y be the state variables, and let $P_0 = (X_0, Y_0)$ be the initial condition.

For each $i \geqslant 0$,

$$(X_{i+1}, Y_{i+1}) = (X_i, Y_i) + \Delta t \cdot (X'(X_i, Y_i), Y'(X_i, Y_i)).$$

What I am hoping to see here is an explicit recognition that points have two coordinates and vectors have two components.

Nevertheless, any reasonable statement of Euler's method: 2 pts.

b) (8 pts) Recall the "Shark-Tuna" model of a predator-prey system:

$$\begin{cases} S' = ST - S \\ T' = T - ST \end{cases}$$

where S represents the number of sharks (predators) and T represents the number of tuna (prey).

Suppose that a system modeled by the Shark-Tuna model starts at an initial state, at time $t = 0$ years, in which there are 2 sharks and 3 tuna. Use Euler's method, with a time increment $\Delta t = 0.1$ years, to approximate the numbers of sharks and tuna at time $t = 0.2$ years.

Solution At $t = 0.1$, $S \approx 2.4, T \approx 2.7$; at $t = 0.2$, $S \approx 2.808, T \approx 2.322$.

Each of S, T at each stage: 2 pts

Full partial credit may be given for a relatively correct second stage answer based on an incorrect first stage answer.

6. a) (1 pt) State the definition of the **derivative of** $F = F(X)$ **with respect to** X **at** $X = a$:

Solution

$$\left.\frac{dF}{dX}\right|_{X=a} = \lim_{\Delta X \to 0} \frac{F(a + \Delta X) - F(a)}{\Delta X}$$

The limit and the difference quotient: 1 pt each

b) (1 pts) Explain how the definition you gave in part a) is related to the average rate of change of F near the point $(a, F(a))$.

Solution The derivative of $F(X)$ at $X = a$ is defined as the limit of the difference quotient as ΔX approaches 0. The difference quotient, in turn, represents the average rate of change of $F(X)$ near the point $(a, F(a))$. More precisely, it represents the average rate of change of $F(X)$ over the interval from $X = a$ to $X = a + \Delta X$. Thus, the derivative is defined as the limit of the average rates of change as ΔX approaches 0, and may be taken to be the definition of *instantaneous rate of change* of $F(X)$ at $X = a$.

Any reasonable rephrasing of the above: 1 pt

c) (2 pts) State both forms of the formula for linear approximation in terms of $F(X)$ and X.

Solution Let $\Delta X = b - a$ and $\Delta F = F(b) - F(a)$. Then,

1) $\Delta F \approx \left.\frac{dF}{dX}\right|_{X=a} \Delta X$;

2) $F(b) \approx F(a) + \left.\frac{dF}{dX}\right|_{X=a} \Delta X$

Each form: 1 pt

d) (3 pts) By using **only** the definition you stated in part *a*) of this problem, find the derivative of

$$F(X) = 3X^2 + 4X + 5$$

with respect to X at $X = 1$.

Solution

$$\left.\frac{dF}{dX}\right|_{X=1} = \lim_{\Delta X \to 0} \frac{F(1+\Delta X) - F(1)}{\Delta X}$$

$$= \lim_{\Delta X \to 0} \frac{(3(1+\Delta X)^2 + 4(1+\Delta X) + 5) - (3(1)^2 + 4(1) + 5)}{\Delta X}$$

$$= \lim_{\Delta X \to 0} \frac{3 + 6\Delta X + 3(\Delta X)^2 + 4 + 4\Delta X + 5 - 3 - 4 - 5}{\Delta X}$$

$$= \lim_{\Delta X \to 0} \frac{6\Delta X + 3(\Delta X)^2 + 4\Delta X}{\Delta X}$$

$$= \lim_{\Delta X \to 0}(10 + 3\Delta X)$$

$$= 10$$

Correct set-up: 1 pt; simplifications and cancellations: 1 pt; correct numerical answer: 1 pt

e) (3 pts) By using your answers to parts *c*) and *d*) of this problem, use linear approximation to estimate $F(1.1)$, where $F(X)$ is the polynomial from part *d*).

Solution

$$F(1.1) \approx F(1) + \left.\frac{dF}{dX}\right|_{X=1}(1.1 - 1)$$

$$= 12 + (10)(0.1)$$

$$= 13.$$

(The actual value is 13.03, so pretty good.)

Correct set-up (including the substitutions): 2 pts, correct final answer: 1 pt