1. (10 points) A protein called NF-κB is a major factor in inflammatory responses in humans. It normally is found in the cytoplasm of your cells, bound to another protein called IκBα. When the cell receives an inflammatory signal, IκBα is degraded. This reveals a special spot on NF-κB that allows it to be imported into the nucleus. Once in the nucleus, NF-κB acts as a "transcription factor" and turns on genes involved in the inflammatory response.

We want to make a dynamical model of this process. We will use the following variables:

$$\begin{aligned} I &= \text{amount of free I}\kappa\text{B}\alpha \\ C &= \text{amount of free NF-}\kappa\text{B in the } \textbf{cytoplasm} \\ N &= \text{amount of free NF-}\kappa\text{B in the } \textbf{nucleus} \\ B &= \text{amount of NF-}\kappa\text{B } \textbf{bound} \text{ to I}\kappa\text{B}\alpha \end{aligned}$$

Here are the assumptions of the model:

- Free IκBα molecules (I) are produced at a constant rate q.
- Free IκBα degrades at a per-molecule rate d.
- Free IκBα molecules (I) can bind with free NF-κB in the cytoplasm (C). This protein interaction only occurs when free IκBα collides with free NF-κB in the cytoplasm, which happens with a rate constant of k. This collision turns the two original molecules into one molecule of the IκBα/NF-κB complex (B).

$I + C \xrightarrow{k} B$

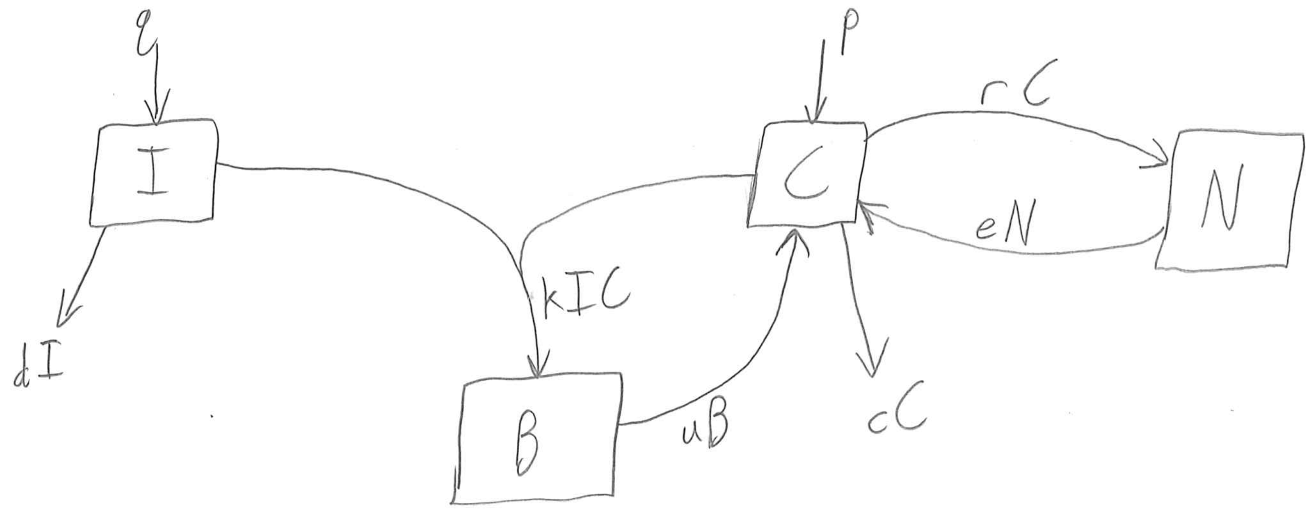
- While bound to NF-κB, the IκBα can degrade, meaning that the complex (B) degrades back into a free NF-κB molecule (C), and the IκBα is lost. This happens at a per-molecule rate of u.
- Free NF-κB molecules (C) are produced in the cytoplasm at a constant rate p.
- Free NF-κB in the cytoplasm degrades at a per-molecule rate c.
- Free NF-κB is imported from the cytoplasm into the nucleus (C $\longrightarrow$ N) at a per-molecule rate of r.
- NF-κB in the nucleus (N) is not produced directly, nor does it degrade. However, it is exported back out to the cytoplasm (N $\longrightarrow$ C) at a per-molecule rate of e.

$C \underset{e}{\overset{r}{\rightleftarrows}} N$

Write a set of differential equations for this model. It is recommended that you start with a diagram.

Certain things here can be thought of just like chemical reactions. (The binding of I and C into B is a chem. reaction.)

Question 1 continues on the next page...



$$\begin{cases} I' = q - dI - kIC \\ C' = p - cC - kIC + uB - rC + eN \\ N' = rC - eN \\ B' = kIC - uB \end{cases}$$

2. (a) (4 points) State the Fundamental Existence and Uniqueness Theorem for differential equations (a.k.a. FTEUSODE, or the Picard–Lindelöf Theorem).

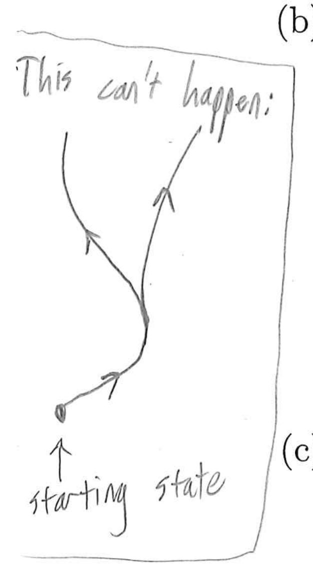
Roughly speaking, it says:

For every state in the state space, there is one and only one trajectory that passes through that state.

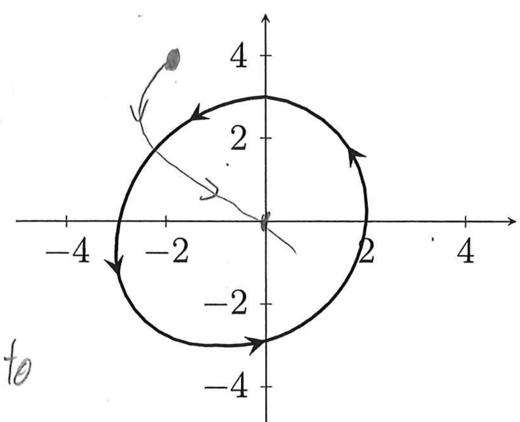
(b) (3 points) Define what it means for a dynamical system to be *deterministic*. Are differential equation models deterministic? Explain briefly why or why not.

Determinism means that the initial state completely determines the entire future trajectory of the system.

Yes, differential equation models are deterministic. This is because, thanks to FTEUSODE, there is only one trajectory that can proceed from the initial state point. (See picture at left.)



(c) (3 points) The figure to the right shows a trajectory for a two-variable system of differential equations. Suppose another trajectory starts at the state $(-2, 4)$. Could this other trajectory pass through the origin, $(0, 0)$? Why or why not?



No, this cannot happen, because it would have to cross the other trajectory. And thanks to the Existence and Uniqueness Theorem, trajectories cannot intersect each other.

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3. (10 points) Red-tailed hawks prey on squirrels on the campus of UCLA. The following differential equations model the local populations of the hawks (H) and squirrels (S):

$$\begin{cases} H' = 0.1SH - 0.5H \\ S' = 0.1S - \frac{SH}{10+S} \end{cases}$$

Suppose that initially, at time $t = 0$, there are 3 hawks and 20 squirrels. Using a step size of $\Delta t = 0.1$, find the (approximate) population sizes at time $t = 0.2$. Use Euler's Method!

t	State $\begin{bmatrix} H \\ S \end{bmatrix}$	change vector $\begin{bmatrix} H' \\ S' \end{bmatrix}$	"next" state
0	$\begin{bmatrix} 3 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 4.5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 20 \end{bmatrix} + 0.1 \begin{bmatrix} 4.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.45 \\ 20 \end{bmatrix}$
0.1	$\begin{bmatrix} 3.45 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 5.175 \\ -0.3 \end{bmatrix}$	$\begin{bmatrix} 3.45 \\ 20 \end{bmatrix} + 0.1 \begin{bmatrix} 5.175 \\ -0.3 \end{bmatrix} = \begin{bmatrix} 3.9675 \\ 19.97 \end{bmatrix}$
0.2	$\begin{bmatrix} 3.9675 \\ 19.97 \end{bmatrix}$		

$$\boxed{\begin{array}{ll} H = 3.9675 & \text{(hawks)} \\ S = 19.97 & \text{(squirrels)} \end{array}}$$

Scratch work:

$H=3,\ S=20$: $H' = 0.1 \cdot (3) \cdot (20) - 0.5(3) = 4.5$

$S' = 0.1 \cdot (20) - \frac{(20)\cdot(3)}{10+20} = 2 - 2 = 0$

$H=3.45,\ S=20$: $H' = 0.1 \cdot (3.45) \cdot (20) - 0.5 \cdot (3.45) = 6.9 - 1.725 = 5.175$

$S' = 0.1 \cdot (20) - \frac{(20)\cdot(3.45)}{10+20} = 2 - 2.3 = -0.3$

4. You are managing a large hamster colony. You started your colony with 10 hamsters. The size of the population, $H(t)$, is given by:

$$H(t) = \frac{200e^t}{19 + e^t}$$

Here the time variable t is in months.

(a) (5 points) Calculate the **average rate of increase** of the population from $t = 4$ to $t = 6$ months. Do the same thing for the average growth of the population from $t = 4$ to $t = 5$ months. Calculate this also for $t = 4$ to $t = 4.5$ months.

From 4 to 6: $\frac{H(6)-H(4)}{6-4} = \frac{191.00 - 148.37}{6-4}$

$= \boxed{21.32 \text{ hamsters/month}}$

From 4 to 5: $\frac{H(5)-H(4)}{5-4} = \frac{177.30 - 148.37}{5-4}$

$= \boxed{28.93 \text{ hamsters/month}}$

From 4 to 4.5: $\frac{H(4.5)-H(4)}{4.5-4} = \frac{165.14 - 148.37}{4.5-4}$

$= \boxed{33.55 \text{ hamsters/month}}$

Question 4 continues on the next page...

(b) (4 points) Compute the *symbolic derivative* $H'(t)$. *(Hint: Don't forget the quotient rule!!)* You don't need to fully expand the denominator.

$$H'(t) = \frac{(200e^t)\cdot(19+e^t) - (200e^t)\cdot(e^t)}{(19+e^t)^2}$$

$$= \frac{200e^t\cdot 19 + \cancel{200e^t\cdot e^t} - \cancel{200e^t\cdot e^t}}{(19+e^t)^2}$$

$$= \boxed{\frac{3800e^t}{(19+e^t)^2}}$$

(Simplified. It's okay to leave it as.)

(c) (3 points) Using your answer from (b), compute $H'(4)$.
(Note: remember, this should be a *number*!).
Explain how this is related to the rates you calculated in part (a) above.

$$H'(4) = \frac{3800e^4}{(19+e^4)^2} = \boxed{38.30 \text{ hamsters/month}}$$

This is the instantaneous rate of change of the hamster population at $t=4$ months, which is what the average rates in part (a) are converging to. I.e., if you look at the average rate from 4 to 4.1, or 4 to 4.001, or 4 to 4.000001, those average rates will converge to 38.30 hamsters/month.

5. Most animals grow to a certain size and stop growing. Some animals, like the great white shark, continue growing for their entire adult lives (at least, for quite a long time). We can make a simple model for this never-ending shark growth. Let's say the age of the shark is t years. Between the ages of 5 and 80 years, we can model the length of the shark in meters as:

$$L = 1.71 \ln(t)$$

Use a linear approximation to this function to answer the following questions:

(a) (2 points) What is the symbolic derivative of the length of the shark with respect to its age. In other words, what is dL/dt?

$$\frac{dL}{dt} = 1.71 \cdot \frac{1}{t} = \boxed{\frac{1.71}{t}}$$

(b) (4 points) What is the value of dL/dt evaluated at the age $t = 10$? Using a linear approximation, predict how much a 10-year-old shark will grow (e.g. ΔL) in the next *half year.*

$$\left.\frac{dL}{dt}\right|_{t=10} = \frac{1.71}{10} = \boxed{0.171 \text{ meters/year}}$$

Linear approx: $\Delta L \approx 0.171 \Delta t$

So $\Delta L \approx 0.171 \cdot (0.5) = \boxed{0.0855 \text{ meters}}$

(8.55 cm)

Question 5 continues on the next page...

(c) (4 points) Use the same method as in (b) to predict how much a 50 year old shark will grow in *half a year.*

$$\left.\frac{dL}{dt}\right|_{t=50} = \frac{1.71}{50} = 0.0342 \text{ meters/year}$$

$$\Delta L \approx 0.0342 \cdot \Delta t \quad \text{(linear approximation)}$$

$$\text{So } \Delta L \approx 0.0342 \cdot (0.5) = \boxed{0.0171 \text{ meters}}$$

(1.71 cm)

(d) (2 points) Compare your answers from (b) and (c). What is happening to the growth rate of the shark as it ages?

As it gets older, it is still growing (positive growth rate), but the growth rate is decreasing.
So the result is that its growth is slowing down.

6. It is important to know the density of wood for construction, manufacturing, etc. The density, D (measured in kg/m^3), of wood in a tree, is a function of the thickness of rings in a tree, R, measured in mm. The thickness of the rings in a tree depends on the amount of water supplied to the tree, W, measured in cm of rainfall in a given year. Say their relationships are given by the following two equations:

$$D(R) = 100e^{-R} + 300$$
$$R(W) = \sqrt{W} + 1$$

(a) (2 points) Say we are interested in measuring the density of wood as a function of rainfall. How would you write this as a function? Your answer should be in terms of W.

$$D(R(W)) = 100e^{-R(W)} + 300 = \boxed{100e^{-(\sqrt{W}+1)} + 300}$$

Composition!

(b) (5 points) What is the derivative of D with respect to W? Your answer should be in terms of W.

One way: $\boxed{\frac{dD}{dW} = \underbrace{100e^{-(\sqrt{W}+1)} \cdot \left(-\frac{1}{2}W^{-\frac{1}{2}}\right)}_{\text{Chain Rule!!!}}}$ (Just differentiating this.)

Another way:

$$\frac{dD}{dW} = \frac{dD}{dR} \cdot \frac{dR}{dW} = \left(-100e^{-R}\right) \cdot \left(\frac{1}{2}W^{-\frac{1}{2}}\right)$$

Need to write this in terms of W

$$= \boxed{\left(-100e^{-(\sqrt{W}+1)}\right) \cdot \left(\frac{1}{2}W^{-\frac{1}{2}}\right)}$$

Chain Rule!!!

Question 6 continues on the next page...

$$= \boxed{\frac{-50e^{-(\sqrt{W}+1)}}{\sqrt{W}}}$$ (simplified)

(c) (3 points) Say the derivative $\frac{dD}{dW} = 0$ at some value of W. What would happen if you increase W a little bit at this point?

Linear approximation says $\Delta D \approx 0 \cdot \Delta W$ in this case. So the change in D would be approximately 0. So, in other words, the tree density would barely change at all.

7. (8 points) Say you have the following code:

```
def avgRateOfChange(f, x1, x2):
    f1 = f(x1)
    f2 = f(x2)
    change_in_f = f2 - f1
    change_in_x = x2 - x1
    return change_in_f / change_in_x
```

Let $f(x) = \sin(x)$. You want to **compute a list of closer and closer approximations to $f'(2)$, and print this list**. On the following page, there are seven lines of code. Specify a *correct order* for these lines, and which ones need to be *indented*, to accomplish this.

Leave your answer in the box below:

```
A
B
E
G
        F
        D

C
```

These first three could be in any order. Also, line E could be right after G, indented (i.e. inside the for loop)

Note that you don't have to write out the code, just the letters of the lines. For example, you could answer something like the following:

```
A
B
    C
    D
E
F
G
```

Question 7 continues on the next page...

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Lines of code:

A. `x_list = [5, 4, 3, 2.5, 2.1, 2.01]`

B. `output_list = []`

C. `print(output_list)`

D. `output_list.append(this_avg_rate)`

E. `f(x) = sin(x)`

F. `this_avg_rate = avgRateOfChange(f, 2, x2)`

G. `for x2 in x_list:`