Statistics 105 Midterm Exam 1 (April 30, 2012)

(Total 4 problems and 30 points)

Name ______________ UID ______________ Discussion 1B Score 24-1

1. (12 points) Answer the question **briefly** or fill in the blanks.

   8

   (a) Let X and Y be two independent random variables with $E(X) = 1, V(X) = 3, E(Y) = 3$, and $V(Y) = 4$. Then

   $E(2X - 3Y + 1) =$ -6 $2(1) - 3(3) + 1$

   $V(2X - 3Y + 1) =$ 48 $2^2(3) + (-3)^2 4 + 0$

   (b) A random sample of $n = 16$ results in $\bar{x} = 13$ and $s = 5$. An estimate of the population mean μ is 13 and its estimated standard error is 1.25

   (c) Let X_1, X_2, X_3, X_4 be a random sample from a normal distribution with mean 4 and variance 9. What is sampling distribution of sample mean $\bar{X}$?

   $E(\bar{x}) = \frac{1}{4}\sum_{i=1}^{4} E(X_i) = \frac{1}{4}\sum \mu = \mu = 4$

   $V(\bar{x}) = \frac{1}{4^2}\sum_{i=1}^{4} V(X_i) = \frac{1}{4^2}\sum \sigma^2 = \frac{\sigma^2}{4} = \frac{9}{4}$ $\bar{X} \sim N(4, 9/4)$

   (d) When we make a normal probability plot for data from a normal population, we expect

   a straight line from bottom left to top right

   (e) For a normal random variable X with mean 5 and variance 4, find the 85th percentile, that is, find x such that $P(X \le x) = 0.85$.

   using TI-83:

   invNorm(0.85, 5, 2) = 7.07

   (f) Let X_1, X_2, X_3, X_4 be a random sample from a normal distribution with mean 3 and variance 16. Find $P(\bar{X} > 2)$.   Var = 4   SD = 2

   $\bar{x} \sim N(3, 16/4) = N(3, 4)$

   TI-83: normalcdf(2, 1E99, 3, 4) = 0.6997

   -1

2. (4 points) Let $X_1, \ldots, X_4$ be a random sample of size $n = 4$ from a population with mean $\theta/2$ and variance θ^2.

   (a) Find the moment estimator $\boxed{\text{of } \theta}$.

   $\bar{x} = \mu$ $\hat{\mu} = \bar{X}$

   $\sigma^2 + \mu^2 = \sum_{i=1}^{n} X_i^2$ $\hat{\sigma}^2 = \cancel{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}}$

   -2

   (b) What are the mean and variance of sample mean $\bar{X}$?

   $\mu = \bar{x} = \theta/2$

   $\hat{\sigma}^2 = \cancel{\frac{\sum_{i=1}^{4}(X_i - \theta/2)^2}{4}}$ ok

   -1

   $V(\bar{X}) = ?$

3. (4 points) Clearly circle your answer for each statement.

(a) Which measure of location is resistant to outliers? (circle one) **sample mean** or **sample median**

(b) For the normal population $N(\mu, \sigma^2)$, the sample mean $\bar{X}$ is an unbiased estimator of μ and has the minimum variance among all unbiased estimators. (circle one) **True** or **False**

(c) For the normal population $N(\mu, \sigma^2)$, the maximum likelihood estimator of the variance σ^2 is not unbiased. (circle one) **True** or **False**

(d) An unbiased estimator always has smaller mean square error than a biased estimator. (circle one) **True** or **False**

4. (10 points) Suppose X has probability density function

$$f(x, \theta) = \left(\frac{2\theta}{\pi}\right) e^{-\theta^2 x^2/\pi}, \text{ for } x > 0$$

$E(x) = 1/\theta \qquad V(x) = (\pi/2 - 1)\frac{1}{\theta^2}$

where $\theta > 0$ is an unknown parameter.

(a) Find the maximum likelihood estimator of θ based on a random sample of size n. Show your work.

$L(\theta) = \left(\frac{2\theta}{\pi}\right) e^{-\frac{\theta^2 x_1^2}{\pi}} \cdot \left(\frac{2\theta}{\pi}\right) e^{-\frac{\theta^2 x_2^2}{\pi}} \cdot \ldots \cdot \left(\frac{2\theta}{\pi}\right) e^{-\frac{\theta^2 x_n^2}{\pi}}$

$= \left(\frac{2\theta}{\pi}\right)^n e^{-\frac{\theta^2}{\pi}\sum_{i=1}^n x_i^2}$

$\ln(L(\theta)) = n \ln\left(\frac{2\theta}{\pi}\right) - \frac{\theta^2}{\pi}\sum_{i=1}^n (x_i)^2$

$\frac{d}{d\theta}(\ln(L(\theta))) = \frac{n\cancel{\pi}}{\cancel{2}\theta} - \frac{2\theta}{\pi}\sum_{i=1}^n (x_i)^2 = 0$

-1

$n\pi^2 = 4\hat{\theta}^2 \sum_{i=1}^n (x_i)^2$

$\hat{\theta} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{n}{\sum_{i=1}^n (x_i)^2}}$

(b) Find the Fisher information $I(\theta)$ of parameter θ. Show your work.

$\frac{d^2}{d\theta^2}(\ln(L(\theta))) = -\frac{n\pi}{2\theta^2} - \frac{2}{\pi}\sum_{i=1}^n (x_i)^2$

$-E\left(\frac{d^2}{d\theta^2}(\ln(L(\theta)))\right) = E\left(\frac{n\pi}{2\theta^2} + \frac{2}{\pi}\sum_{i=1}^n (x_i)^2\right)$

$= \frac{n\cancel{\pi}}{\cancel{2}\theta^2} + \frac{2n}{\pi\cancel{\theta^2}}$

-3

-2

$= \frac{n}{2\pi\theta^2}(\pi^2 + 4)$

Statistics 105 Midterm Exam Solution (Spring 2012)

1. (a) $E(2X-3Y+1) = (2)(1) - (3)(3) + 1 = -6$, $V(2X-3Y+1) = (2^2)(3) + (3^2)(4) = 48$.
   (b) $\bar{x} = 13$ and $s/\sqrt{n} = 5/4 = 1.25$
   (c) $\bar{X} \sim N(\mu = 4, \sigma^2/n = 9/4)$
   (d) all points close to a straight line.
   (e) $z = 1.04$ so $x = \mu + z\sigma = 5 + (1.04)(2) = 7.08$.
   (f) $\bar{X} \sim N(\mu = 3, \sigma^2/n = 16/4)$. $P(\bar{X} > 2) = P(Z > (2-3)/\sqrt{16/4}) = P(Z > -0.5) = 1 - 0.3085 = 0.6915$

2. (a) Let $\bar{X} = \theta/2$ and solve for θ. The moment estimator is $\hat{\theta} = 2\bar{X}$.
   (b) $E(\bar{X}) = \theta/2$ and $V(\bar{X}) = \sigma^2/n = \theta^2/4$.

3. (a) sample median; (b) True; (c) True; (d) False.

4. (a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \left(\frac{2\theta}{\pi}\right) e^{-\theta^2 x_i^2/\pi} = \left(\frac{2\theta}{\pi}\right)^n e^{-\frac{\theta^2}{\pi}\sum_{i=1}^{n} x_i^2}$$

$$\ln L(\theta) = n \ln(2\theta) - n \ln \pi - \frac{\theta^2}{\pi}\sum_{i=1}^{n} x_i^2$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} - \frac{2\theta}{\pi}\sum_{i=1}^{n} x_i^2$$

Let $\frac{d \ln L(\theta)}{d\theta} = 0$. We obtain MLE

$$\hat{\theta} = \sqrt{\frac{n\pi}{2\sum_{i=1}^{n} X_i^2}}$$

   (b) $\ln f(x, \theta) = \ln(2\theta) - \ln \pi - \frac{\theta^2}{\pi}x^2$. We have

$$\frac{\partial \ln f(x,\theta)}{\partial \theta} = \frac{1}{\theta} - \frac{2\theta}{\pi}x^2 \text{ and } \frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2} = -\frac{1}{\theta^2} - \frac{2}{\pi}x^2.$$

$$I(\theta) = -E\left(\frac{\partial^2 \ln f(X,\theta)}{\partial \theta^2}\right) = -E\left(-\frac{1}{\theta^2} - \frac{2}{\pi}X^2\right) = \frac{1}{\theta^2} + \frac{2}{\pi}E(X^2) = \frac{1}{\theta^2} + \frac{2}{\pi}\frac{1}{\theta^2}\frac{\pi}{2} = \frac{2}{\theta^2}$$

because $E(X^2) = V(X) + (E(X))^2 = \frac{1}{\theta^2}\left(\frac{\pi}{2} - 1\right) + \left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}\frac{\pi}{2}$.

Summary of the scores: n=80, min=8, q1=21, q2=24, q3=27, max=30, mean=23, sd=4.3