STAT 100A MIDTERM EXAM

Notes:

(1) There are 3 problems; each problem has 10 points.

(2) Please show all the necessary steps in your answers, and please write your answers with precise notation and coherent English. If there is not enough space, please use the reverse side of the page.

Your name: [redacted]

Your ID [redacted]

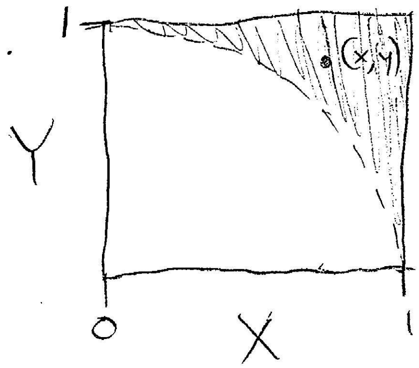
Problems	No. 1	No. 2	No. 3	Total
Scores	10	10	10	

Problem 1: Suppose we generate two random numbers X and Y from Uniform[0,1] independently, so that (X, Y) is a random point in the unit square $[0, 1]^2$.

(1) Calculate $P(X^2 + Y^2 > 1)$.

(2) Calculate $P(X < 1/2 | Y > X)$. (Hint: within the unit square, the region $y > x$ is a triangle)

1.



As seen on the diagram on the left, the area of the region of $X^2+Y^2>1$ is the entire square area minus the area of the region $X^2+Y^2 \leq 1$. Thus:

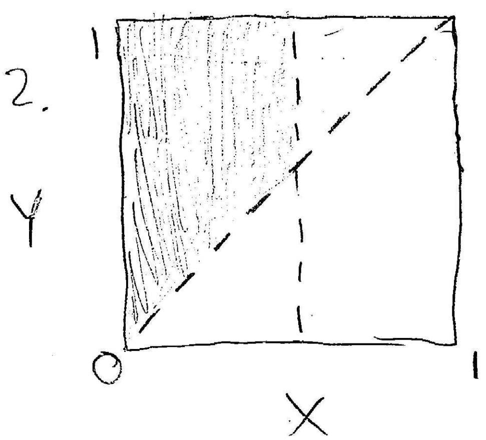
$$P(X^2+Y^2>1) = 1 - P(X^2+Y^2 \leq 1)$$

$$= 1 - \frac{\frac{1}{4}(\pi)(1^2)}{1(1)} = 1 - \frac{\pi}{4}$$

$$= \frac{4-\pi}{4}$$

✓

2.



As seen on the diagram on the left:

$$P(X < ½ | Y > X) = \frac{P(X < ½ \cap Y > X)}{P(Y > X)}$$

$$= \frac{3/8}{1/2} = \frac{3}{4}$$

✓

$=E\left(\left(\frac{X}{n}\right)^2\right)$

Problem 2: Suppose we flip a coin $n = 100$ times independently. The probability of getting a head is $p = .5$ in each flip. Let X be the number of heads.

(1) What are E(X) and Var(X)?

(2) What are E(X/n) and Var(X/n)?

(3) What is $P(40 \le X \le 60)$? You only need to write down the formula in terms of binomial probabilities without calculating the number.

$E\left(\left(\frac{X}{n} - E\left(\frac{X}{n}\right)\right)^2\right) = E\left(\left(\frac{X}{n} - \frac{1}{2}\right)^2\right)$

1) $E(X) = np = 100\left(\frac{1}{2}\right) = \boxed{50 = E(X)}$ ✓

$Var(X) = np(1-p) = 100\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \boxed{25 = Var(X)}$ ✓

2) $E\left(\frac{X}{n}\right) = \frac{1}{n}(E(X)) = \frac{1}{100}(50) = \boxed{\frac{1}{2} = E\left(\frac{X}{n}\right)}$ ✓

$Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X) = \frac{1}{100^2}(25) = \boxed{\frac{1}{400} = Var\left(\frac{X}{n}\right)}$ ✓

3) $P(40 \le X \le 60) = \sum_{k=40}^{60} \binom{100}{k} \left(\frac{1}{2}\right)^k \left(1-\frac{1}{2}\right)^{100-k}$

$\boxed{\sum_{k=40}^{60} \binom{100}{k} \left(\frac{1}{2}\right)^{100}}$ ✓

Problem 3: Suppose X is a discrete random variable following distribution $p(x)$, where x takes values in a discrete set.

(1) Prove $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$.

(2) Prove $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.

(3) Let $\mu = \mathrm{E}(X)$, and $\sigma^2 = \mathrm{Var}(X)$. Let $Z = (X-\mu)/\sigma$. Calculate $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.
$=0$ $=1$

1) $E(aX+b) = \sum_x (ax+b)p(x)$

$= \sum_x ax\,p(x) + b\,p(x)$

$= a\sum_x x\,p(x) + b\sum_x p(x) = aE(X) + bE(1)$

$= aE(X) + b$ ■

2) $Var(aX+b) = E\left((aX+b-E(aX+b))^2\right)$

$= E\left((aX+b-(aE(X)+b))^2\right)$

$= E\left((aX+\cancel{b}-aE(X)-\cancel{b})^2\right)$

$= E\left(a^2(X-E(X))^2\right) = a^2E\left((X-E(X))^2\right)$

$= a^2\,Var(X)$ ■

3) $E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu) = \frac{1}{\sigma}(E(X)-\mu)$

$= \frac{1}{\sigma}(E(X)-E(X)) = \frac{1}{\sigma}(0)$

$= \boxed{0 = E(Z)}$

$Var(Z) = E\left((Z-E(Z))^2\right) = E\left((Z-0)^2\right) = E(Z^2) = E\left(\frac{(X-E(X))^2}{Var(X)}\right)$

$= E\left(\frac{Var(X)}{Var(X)}\right) = E(1) = \boxed{1 = Var(Z)}$