STAT 100A Midterm

Notes:

(1) There are 3 problems; each problem has 10 points.

(2) Please use precise notation, and show all the necessary steps in your calculations. If there is not enough space, please use the reverse side of the page.

Your name [redacted]
Your ID: [redacted]

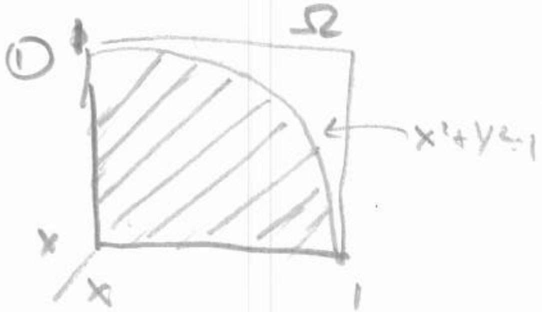
Problems	No. 1	No. 2	No. 3	Total
Scores	10	10	10	30

Problem 1: Suppose we generate two independent random variables X and Y uniformly over $[0, 1]$.

(1) (4 points) Calculate $P(X^2 + Y^2 \leq 1)$.

(2) (6 points) Calculate $P(X > 1/2 | X + Y < 1)$.

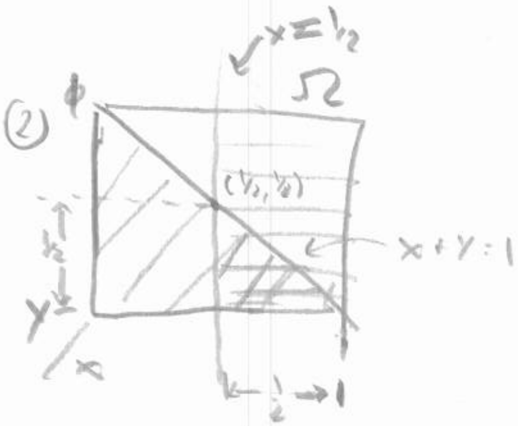
1



$$P(x^2+y^2 \leq 1) = \text{area of shaded region}$$

$$= \frac{1}{4}\pi(1)^2 = \boxed{\frac{\pi}{4}}$$

2



$$P(x>\tfrac{1}{2} \mid x+y<1) = \frac{P(x>\frac{1}{2} \cap x+y<1)}{P(x+y<1)}$$

$$= \frac{\text{area of region shaded by both}}{\text{area of region where } x+y<1}$$

$$= \frac{\frac{1}{2}(\frac{1}{2})(\frac{1}{2})}{\frac{1}{2}(1)(1)} = \frac{\frac{1}{4}}{1} = \boxed{\frac{1}{4}}$$

Problem 2: Suppose 1% of the population is inflicted with a particular disease. For a medical test, if a person has the disease, then 90% chance the person will be tested positive. If a person does not have the disease, then 90% chance the person will be tested negative.

(1) (4 points) What is the probability that a randomly selected person will be tested negative?

(2) (6 points) If the person is tested negative, what is the chance that he or she does not have the disease?

Givens:

$P(D) = 1\%$

$P(+|D) = 90\%$

$P(-|N) = 90\%$

Therefore $P(N) = 1 - P(D) = 99\%$

$P(-|D) = 1 - P(+|D) = 10\%$

$P(+|N) = 1 - P(-|N) = 10\%$

(1) $P(-) = P(- \cap N) + P(- \cap D)$ by the law of total probability

by the chain rule

$= P(-|N)P(N) + P(-|D)P(D) = (90\%)(99\%) + (10\%)(1\%)$

$= \dfrac{90 \cdot 99 + 10 \cdot 1}{100^2} = \dfrac{8910 + 10}{100^2} = \dfrac{8920}{10000} = \dfrac{89.2}{100} = \boxed{89.2\%}$

$$\begin{array}{r} 99 \\ \times 90 \\ \hline 00 \\ 8910 \\ \hline 8910 \end{array}$$

(2) $P(N|-) = \dfrac{P(N \cap -)}{P(-)} = \dfrac{P(- \cap N)}{P(-)} = \dfrac{P(-|N)P(N)}{P(-)}$

Be Careful!

$= \dfrac{\frac{90}{100} \cdot \frac{99}{100}}{\frac{89.2}{100}} = \dfrac{1}{100} \cdot \dfrac{90 \cdot 99}{89.2} = \dfrac{8910}{89.2} \cdot \dfrac{1}{100} = \boxed{\dfrac{89.1}{89.2}} \approx 99\%$

(I do not have a calculator)

Problem 3: Suppose a person is doing a random walk over two states 1 and 2 according to the following scheme. At each step, regardless of his past history, he stays where he is with probability 1/3, and he moves to the other state with probability 2/3. We use X_t to denote the state of the person at time $t = 0, 1, \ldots$ Suppose $X_0 = 1$.

(1) (5 points) Calculate $P(X_1 = 2)$ and $P(X_2 = 1)$.

(2) (5 points) Calculate $P(X_2 = 1 | X_1 = 2)$ and $P(X_1 = 2 | X_2 = 1)$.

$X_0 = 1$ $\quad$ $P(\text{move}) = \frac{2}{3}$

$1 \leftrightarrow 2$ $\quad$ $P(\text{stay}) = \frac{1}{3}$

(1) $P(X_1 = 2) = P(\text{move from } 1 \text{ to } 2) = P(\text{change state}) = \boxed{\frac{2}{3}}$

$P(X_2 = 1) = P(X_2 = 1 \cap X_1 = 1) + P(X_2 = 1 \cap X_1 = 2)$ by the law of total probability

$= P(X_2 = 1 / X_1 = 1) P(X_1 = 1) + P(X_2 = 1 / X_1 = 2) P(X_1 = 2)$ by the chain rule

$= P(\text{stay}) P(\text{stay}) + P(\text{move}) P(\text{move})$

$= \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \boxed{\frac{5}{9}}$

(2) $P(X_2 = 1 / X_1 = 2) = P(\text{move from } 2 \text{ to } 1) = P(\text{move, in general})$

$= \boxed{\frac{2}{3}}$

by the chain rule

$P(X_1 = 2 / X_2 = 1) = \dfrac{P(X_1 = 2 \cap X_2 = 1)}{P(X_2 = 1)} = \dfrac{P(X_2 = 1 \cap X_1 = 2)}{P(X_2 = 1)}$ since $P(A \cap B) = P(B \cap A)$

$= \dfrac{P(X_2 = 1 / X_1 = 2) P(X_1 = 2)}{P(X_2 = 1)} = \dfrac{\frac{2}{3} \cdot \frac{2}{3}}{5/9} = \frac{9}{5} \cdot \frac{4}{9} = \boxed{\frac{4}{5}}$