Physics 1CH – Midterm Exam
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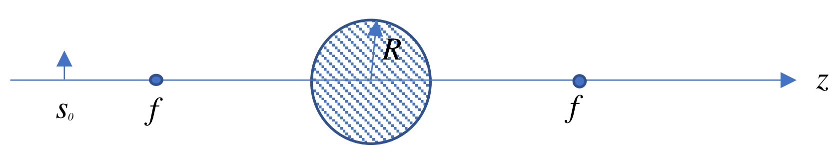
This exam permits one page of notes, both sides, as a primary aid. Please submit a copy of your notes sheet as Problem 5 to receive credit for creating them.
Please submit your exam through the Gradescope portal within 45 minutes of the exam's finish time at 8 PM.
Read each problem carefully; show all your work for full credit.

1. Consider a medium having a common form of a plane wave dispersion relation, $\omega^2 = k^2c^2 + \omega_c^2$, where ω_c^2 is a constant.
(a) (5 pts) What is the phase velocity of a wave with $\omega^2 > \omega_c^2$? (5 pts) Describe the spatial dependence of the electric field if $\omega^2 < \omega_c^2$.
(b) (5 pts) What is the group velocity of a wave with $\omega^2 > \omega_c^2$?
(c) (10 pts) Consider two waves of equal amplitude E_0 superimposed in this medium, one with frequency $\omega = 4\omega_c$ and one with $\omega = 5\omega_c$. Write the superposition of these waves to obtain a beatwave, using the trigonometric relationship $\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$. (5 pts) What is the velocity of propagation of the envelope (the cosine corresponding to the difference in frequencies, not the average)? (5 pts) Compare this to the group velocity at the average of the two frequencies $\omega = 4.5\omega_c$.

2. A plane electromagnetic wave traveling in the +z-direction impinges (from medium 1) *normally* on an interface at z=0 plane, which marks the boundary between two different dielectrics. Both media have $\mu = \mu_0$, and differ only in permittivity, ε_1 and $\varepsilon_2 > \varepsilon_1$, respectively. The electric field polarization is in the x-direction.
(a) Consider the incident wave only. (5 pts) What is the ratio of electric to magnetic field? (5 pts) What is the Poynting vector associated with this incident wave? (10 pts) What is the relationship between the Poynting vector, the electromagnetic energy density and the wave (phase) velocity in the medium?
(b) (15 pts) Develop the (Fresnel) relations for the transmitted and reflected wave electric field amplitudes relative to the incident wave amplitude at this interface.
(c) (10 pts) What are the Poynting vectors associated with the reflected and transmitted wave? How are they related to the incident wave Poynting vector?

3. A dielectric sphere ($\varepsilon = 1.5\varepsilon_0$, radius R) is used as a focusing lens, as shown below:



(a) (5 pts) For paraxial rays, what is the approximate thin lens focal length f?
(b) (5 pts) For the condition given, with $s_o = \frac{3f}{2}$, where is the image? (5 pts) Draw the ray diagram. (5 pts) Is the image real or virtual?

(c) (10 pts) Now consider the lens to be thick. Sketch two rays from the object tip, one a parallel paraxial ray (offset much smaller than R) and one that goes through the center of the sphere. Does the image move closer to the sphere or farther?

(d) (15 pts) Now analyze the position of the image using either of two methods: the matrix method presented in lecture, or sequential application of the formula

$$\frac{1}{s_0} + \frac{1}{s_i} = \frac{1}{f_s}$$

where you consider the focal length of each surface f_s of the sphere (*i.e.* that towards the object, and that towards the image) separately.

4. We would like to propagate a cylindrical laser beam from air $n \simeq 1$ into glass $n = 1.5$ at an angle, so that no power is reflected.
   (a) (5 pts) What polarization should we choose?
   (b) (5 pts) What angle of incidence should we choose?
   (c) (10 pts) What is the shape of the transmitted laser beam in the glass?

5. Submit your notes page, both sides. (5 pts)

Midterm Solutions

$\omega = W$

Problem 1.

(a) $\omega^2 = k^2c^2 + \omega_c^2$ ← const.

ω_c: cutoff

$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_c}{c}\right)^2} \Rightarrow v_g = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$

$(\omega^2 > \omega_c^2)$

if $\omega^2 < \omega_c^2$ $\quad k = i\frac{\sqrt{\omega_c^2 - \omega^2}}{c} = i\kappa$ ← real.

and taking nominal prop. in z-direction we have fields $E_i \sim e^{ikz} \sim e^{-\kappa z}$ ← attenuating

$e^{i(kz-\omega t)}$, $\cos(kz - \omega t)$, $\cos k\left(z - \frac{\omega}{k}t\right)$

(b) Group velocity

$v_g = \frac{d\omega}{dk} = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} < c$ $\qquad \omega^2 > \omega_c^2$

(c) $\omega_1 = 4\omega_c \qquad \omega_2 = 5\omega_c$

$k_1 = \sqrt{15}\frac{\omega_c}{c} \qquad k_2 = \sqrt{24}\frac{\omega_c}{c}$

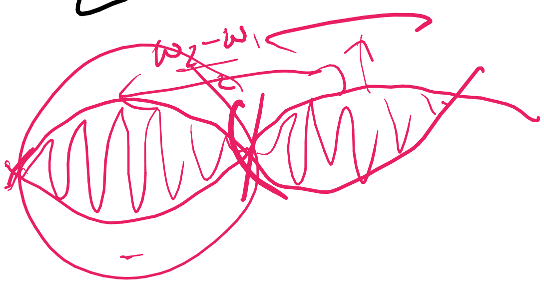
$\Rightarrow E_1 = E_0 \cos\left(\frac{\sqrt{15}\,\omega_c}{c}z - 4\omega_c t\right)$

and $E_2 = E_0 \cos\left(\sqrt{24}\frac{\omega_c}{c}z - 5\omega_c t\right)$

Superimposing, we have

$E = E_1 + E_2 = 2E_0 \cos\left(\frac{(\sqrt{15}+\sqrt{24})}{2}\frac{\omega_c}{c}z - \frac{9}{2}\omega_c t\right) \times$

envelope → $\cos\left(\frac{\sqrt{24}-\sqrt{15}}{2}\frac{\omega_c}{c}z - \frac{\omega_c t}{2}\right)$ (kz, ωt; k_2-k_1, $\omega_2-\omega_1$)



$\Rightarrow v_{env} = \frac{\Delta\omega}{\Delta k} = \frac{c}{\sqrt{24}-\sqrt{15}} \simeq 0.975c$

Compare to group velocity at $\omega = 4.5\omega_c$

$\frac{d\omega}{dk} = v_g = c\sqrt{1-\left(\frac{1}{4.5}\right)^2} \simeq 0.975c$ ✓

$\bar{\omega} = \frac{\omega_1+\omega_2}{2}$

$\cos\left[\frac{k_2-k_1}{2}z - \frac{(\omega_2-\omega_1)}{2}t\right]$

$\Delta k \qquad \Delta\omega$

$v_g = \frac{\Delta\omega}{\Delta k}$

Problem 2

(a) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$ plane wave $\vec{k}_1\times\vec{E}_i = \omega\vec{B}_i$

and $|\vec{E}_i| = \frac{\omega}{k_1}|\vec{B}_i| = \frac{c}{n_1}|\vec{B}_i|$ $\quad n_1 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}$

Poynting vector

$$\vec{S}_i = \vec{E}_i\times\vec{H}_i = \frac{n_1}{c}\frac{\vec{E}_i^2}{\mu_0}\hat{z} \qquad \vec{E}_i = E_{0i}e^{i(kz-\omega t)}\hat{x}$$

Extract $v_\varphi = \frac{c}{n_1}$ as factor $\quad v_\varphi = v_g = \frac{c}{n}$

$$\vec{S} = \frac{c}{n_1}\frac{n_1^2}{c^2\mu_0}\vec{E}_i^2\hat{z} = v_\varphi \underbrace{\epsilon E_1^2}_{u_{em}}\hat{z}$$

$\vec{S}$ is the product of $v_\varphi\, u_{em}$ (in prop. direction)

(v_φ = v_g in this case)

(b) Fresnel relations

$D^\perp$, $B^\perp$

$E^{\parallel}$ continuous: $\quad E_{0i} + E_{0r} = E_{0t}$

$H^{\parallel}$ continuous: $\quad \frac{n_1 E_{0i}}{c\mu_0} - \frac{n_1 E_{0r}}{c\mu_0} = \frac{n_2 E_{0t}}{c\mu_0}$

sign flip due to $\vec{k}$ changing direction

Solve for E_{0t} by multiplying 1st eqn by n_1, adding second eq'n.

$$2n_1 E_{0i} = (n_1+n_2)E_{0t}$$

$$\text{or: } t = \frac{E_{0t}}{E_{0i}} = \frac{2n_1}{n_1+n_2}$$

substitute in 1st eqn to give

$$E_{0r} = E_{0t} - E_{0i} = \frac{n_1-n_2}{n_1+n_2}E_{0i} = r E_{0i}$$

$$r = \frac{E_{0r}}{E_{0i}}$$

(c) Poynting vectors

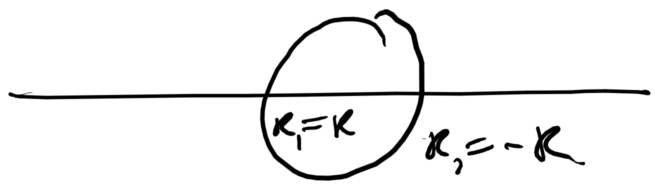
$$\vec{S}_r = -\frac{c}{n_1}\epsilon_1\vec{E}_r^2\hat{z} = -r^2\vec{S}_i$$

$$\vec{S}_t = \frac{c}{n_2}\epsilon_2\vec{E}_t^2\hat{z} = \frac{n_2}{n_1}t^2\vec{S}_i$$

with $R = r^2$, $T = \frac{n_2}{n_1}t^2 \Rightarrow$ $R+T=1$ ✓

Problem 3

(a)



$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$ $\quad n_\ell \simeq 1.22$

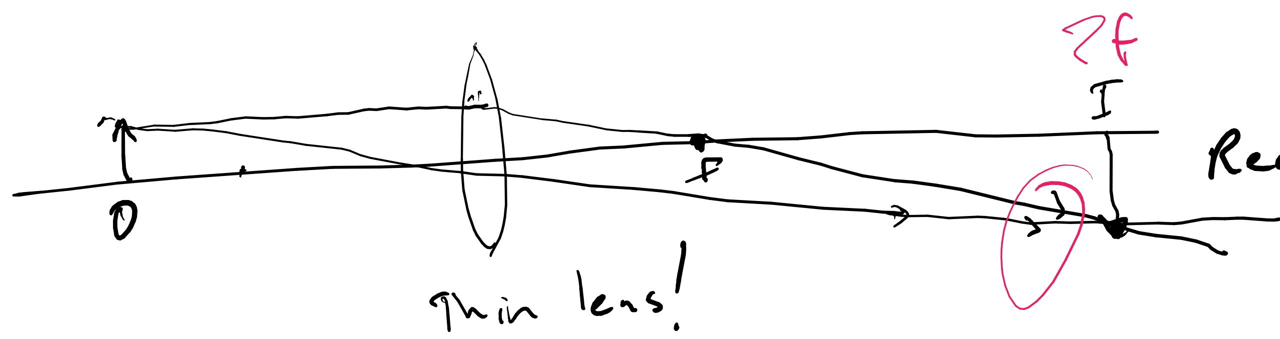
Lens maker's formula

$$\frac{1}{f} = (n_\ell - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{2(n_\ell - 1)}{R} = \frac{0.45}{R}$$

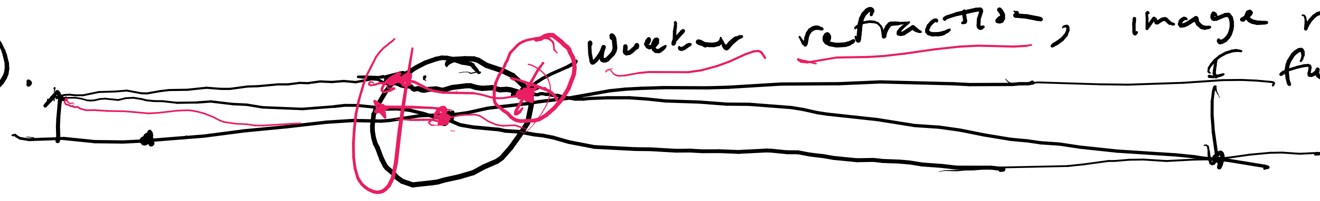
$f \simeq 2.22R$

(b) $\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o}$ $\quad s_o = \frac{3}{2}f$.

$= \frac{1}{f} - \frac{2}{3f} = \frac{1}{3f}$. $\Rightarrow$ $s_i = 3f$.



(c).



(d) Now more quantitatively.
Lets do sequential imaging. Image from surface 1 is as follows

$s_o = \frac{3}{2}f - R$ $\quad f = 4.44R$ ← 2 times thin lens
$= 5.66R$

$M \cdot M_{s_o-s_i} \cdot M_o \cdot M_{s_i} \cdot M_o \neq M_T$
(To image; Lens drif; object – S1)

and $\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} = \frac{1}{4.44R} - \frac{1}{5.26R}$

$s_i \simeq 20.6R$

Imaging? $M_{T,12} \neq 0$ ↳ s_i

Now for surface 2 f is the same and

$s_o = 2R - 20.6R \simeq -18.6R$

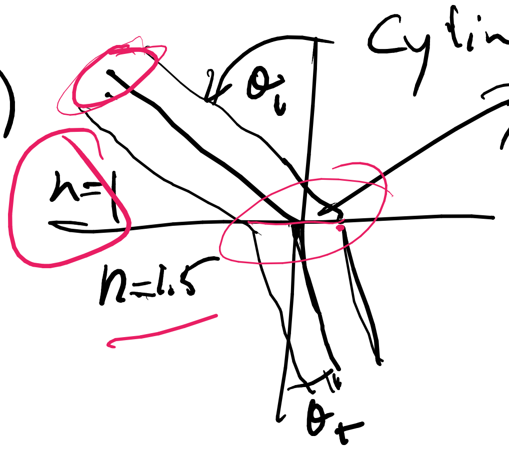
The "second" image is downstream of the final surface by

$\frac{1}{s_i} = \frac{1}{4.44R} + \frac{1}{18.6R}$ and $s_i \simeq 3.58R$.

The image is formed a distance past lens center $\quad s_{i,\text{thick lens}} = s_i + R \simeq 4.56R > 3f$.

S1 offset

Problem 4.

(a) 

Cylindrical beam

No reflection

This is possible only for p-polarization

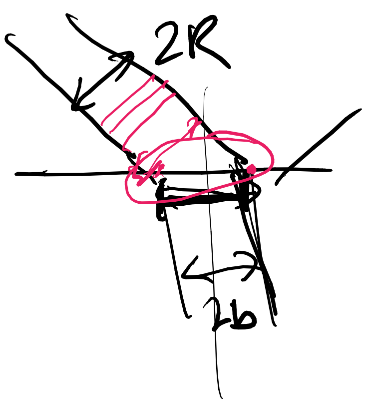
(b). Angle of incidence is Brewster, θ_B

$\tan\theta_i = \tan\theta_B = \frac{n_2}{n_1}$ extinguishes $r^{\parallel}$ (p-pol).

$r^{\parallel}=0$

$\theta_B = 56.3^{\circ}$

(c) The dimensions of the beam in the s-direction are unchanged by the refraction. In the p-direction the beam dimension is changed, and an elliptical shape is obtained in cross-section.



$\frac{2R}{\cos\theta_i}$ at boundary.

$\frac{2b}{\cos\theta_t} = \frac{2R}{\cos\theta_i}$ $\quad b = R\frac{\cos\theta_t}{\cos\theta_i}$

$\cos\theta_i = 0.555$ $\quad \cos\theta_t = 0.832$

$b = 0.666R$ ($\frac{2}{3}$ size)