Problem 1

40 points

Suppose you find or develop a material within which there can be **no magnetic field** (a superconductor, maybe). You send an electromagnetic (EM) plane wave in vacuum towards the surface of this material, with electric field

$$\vec{E}_I = E_I \cos(kz - \omega t)\hat{x}.$$

All answers in this problem should be in terms of the constants E_1, k, ω, and/or c (speed of light in vacuum).

(a): 5 pts

Write down the magnetic field (magnitude & direction) of the incident wave.

$$\vec{B}_I = \frac{E_I}{c}\cos(kz - \omega t)\hat{y}$$

(a): 5 pts

(b): 5 pts

Is there any EM wave inside of the material? If so, write down its electric and magnetic fields.

Since there can be no magnetic field, there can be no EM wave. Any time-varying electric field in the material would incuce an electric field, and so no electric field will exist in the material either.

(b): 5 pts

(c): 8 pts

Write down the electric and magnetic fields (magnitude & direction) of the reflected wave.

The magnetic field needs to vanish on the surface of the material, and so we know the magnetic field must be

$$\vec{B}_R = -\frac{E_I}{c}\cos(kz + \omega t)\hat{y}.$$

Then the electric field must be

$$\vec{E}_R = E_I \cos(kz + \omega t)\hat{x}.$$

We can check: $\hat{x} \times (-\hat{y}) = -\hat{z}$, the direction of propagation.

(c): 8 pts

Problem 1 continued on next page...

(d): 7 pts

Write down the *total* electric and magnetic fields outside the material. You may simplify the result using trig identities.

We can compute:

$$\begin{aligned}\vec{E} &= E_I(\cos(kz+\omega t)+\cos(kz-\omega t))\hat{x}\\ &= 2E_I\cos(kz)\cos(\omega t)\hat{x},\end{aligned}$$

and

$$\begin{aligned}\vec{B} &= \frac{E_I}{c}(\cos(kz-\omega t)-\cos(kz+\omega t))\hat{y}\\ &= 2\frac{E_I}{c}\sin(kz)\sin(\omega t)\hat{y}\end{aligned}$$

(d): 7 pts

(e): 7 pts

Write down the Poynting vector outside the material.

$$\begin{aligned}\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} &= 4\frac{E_I^2}{\mu_0 c}\cos(kz)\cos(\omega t)\sin(kz)\sin(\omega t)\hat{z}\\ &= \frac{E_I^2}{\mu_0 c}\sin(2kz)\sin(2\omega t)\hat{z}\end{aligned}$$

(e): 7 pts

(f): 8 pts

What is the intensity of the wave outside of the material?

The intensity is the magnitude of the time-averaged Poynting vector. The time-average of $\sin(2\omega t)$ is zero over any number of periods, and so

$$I = 0.$$

(f): 8 pts

Problem 2

30 pts

Consider a circuit consisting of a resistor, a capacitor, and an inductor all connected in series, driven by an AC source of fixed magnitude and variable frequency ω. The resistance of the resistor is R, the capacitance of the capacitor is C, and the inductance of the inductor is L. For this problem, you may use without proof any result for impedance derived in lecture, homework, or discussion.

(a): 5 pts

What is the impedance of this circuit? Express your answers in terms of R, L, C, and ω.

We derived the impedance of the LRC series circuit in lecture:

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

(a): 5 pts

(b): 15 pts

At what frequencies ω_+, ω_- will the amplitude of the current through the circuit equal *one-third* the maximum amplitude of current that can flow through the circuit? Express your answers in terms of the given parameters. [HINT 1: What does the impedance need to be for this condition to hold?]. [HINT 2: Make sure your frequencies are positive!]

Recall that the impedance is defined as the proportionality between the amplitudes of current and voltage:

$$V = IZ.$$

Thus for the current amplitude to be one-third of its maximum, the impedance must be three times its minimum value. The minimum of the impedance for the LRC series circuit is $Z_{\text{min}} = R$. Thus we will solve for the frequencies that satisfy $Z(\omega_\pm) = 3R$:

$$\sqrt{R^2 + (\omega L - 1/\omega C)^2} = 3R.$$

Squaring this, we find

$$R^2 + (\omega L - 1/\omega C)^2 = 9R^2.$$

(b): 15 pts

We can rearrange and take the square root. Taking the square root gives us two equations:

$$\omega L - 1/\omega C = \pm\sqrt{8}R.$$

We can rearrange this into a quadratic polynomial:

$$\omega^2 \pm \sqrt{8}\frac{R}{L}\omega - 1/LC = 0.$$

We can use the quadratic equation to solve this:

$$\omega = \sqrt{2}\frac{R}{L}\left(\pm 1 \pm \sqrt{1 + L/2R^2C}\right).$$

Of these four frequencies, two are positive; the two positive frequencies are

$$\omega_+ = \sqrt{2}\frac{R}{L}\left(1 + \sqrt{1 + \frac{L}{2R^2C}}\right)$$
$$\omega_- = \sqrt{2}\frac{R}{L}\left(-1 + \sqrt{1 + \frac{L}{2R^2C}}\right)$$

(b): 15 pts

(c): 10 pts

Describe what changes could be made to R, L, and/or C in order to *double* the maximum possible amount of current flowing through the circuit.

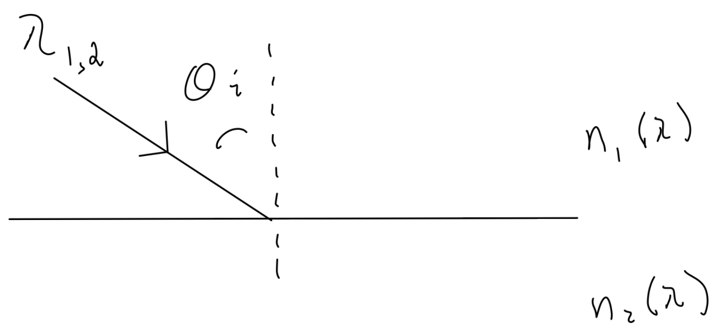
The maximum possible current flows through the circuit at the resonant frequency, at which the impedance is $Z(\omega_{\text{res}}) = R$. Thus changing L and C do not change the maximum possible current, and halving R (and thus $Z(\omega_{\text{res}})$) will double the maximum possible current.

(c): 10 pts

Problem 3

30 pts

Consider two beams of light, of wavelengths λ_1 and λ_2. The light initially travels through some material with index of refraction $n_1(\lambda) = 1 + e^{-\lambda/\lambda_0}$. Both rays are incident on a second material of refractive index $n_2(\lambda) = 2 + e^{-\lambda/\lambda_0}$ at an angle θ_i. Part of the light is reflected, and part of the light is refracted. [Here λ_0 is some fixed wavelength characteristic of the materials.]



(a): 10 pts

At what angle does each beam of light reflect off of the second material?

The angle of reflection does not depend on wavelength or relative index of refraction; thus both beams will reflect off at an angle θ_i.

(a): 10 pts

(b): 10 pts

At what angle does each beam of light refract into the second material?

Snell's law tells us that

$$n_1(\lambda_{1,2}) \sin\theta_i = n_2(\lambda_{1,2}) \sin\theta_{b,1,2}.$$

Thus

$$\theta_{b,1} = \sin^{-1}\left(\frac{n_1(\lambda_1)}{n_2(\lambda_1)}\sin\theta_i\right) = \sin^{-1}\left(\frac{1+e^{-\lambda_1/\lambda_0}}{2+e^{-\lambda_1/\lambda_0}}\sin\theta_i\right)$$

$$\theta_{b,2} = \sin^{-1}\left(\frac{n_1(\lambda_2)}{n_2(\lambda_2)}\sin\theta_i\right) = \sin^{-1}\left(\frac{1+e^{-\lambda_2/\lambda_0}}{2+e^{-\lambda_2/\lambda_0}}\sin\theta_i\right).$$

We don't have to worry about total internal reflection because $n_2 > n_1$ for all λ.

(b): 10 pts

Problem 3 continued on next page. . .

(c): 10 pts

Do the beams bend towards or away from the normal, or does it depend on the specific values of $\lambda_{1,2}$?

Since $n_1(\lambda) < n_2(\lambda)$ for all λ, the beams will always bend towards the normal.	(c): 10 pts