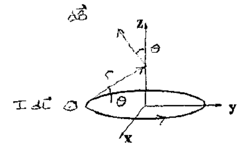


$$d\vec{B} = \frac{\mu_0 I\, d\vec{\ell} \times \hat{r}}{4\pi r^2}$$

$$dB = \frac{\mu_0 I\, ds\, \sin(90^\circ)}{4\pi r^2}$$

$$dB_z = dB \cos\theta$$

$$dB_z = \frac{\mu_0 I R\, ds}{4\pi (R^2 + z^2)^{3/2}}$$

- 1a) (10 points) A circular, conducting loop of radius R lies in the x,y-plane, centered on the origin. A current I flows through the loop such that at $x = +R$, the current is headed in the $+\hat{y}$ direction, and at $x = -R$ the current is headed in the $-\hat{y}$ direction. Derive the resultant magnetic field (magnitude and direction) at every point along the z-axis.

Symmetry: $\vec{B} = B_z \hat{k}$, $B_z = \int dB_z = \frac{\mu_0 I R}{4\pi (R^2+z^2)^{3/2}} \int ds$

$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \hat{k}}$$

← yes, you should have this memorized, I want to see if you can obtain it from BS without BS-ing ☺

- 1b) (10 points) Let's replace the loop with an infinitesimally-thin, uniform ring of electric charge that extends from r to $r + dr$. If the surface charge-density on the ring is given by σ and the ring rotates about the z-axis with a constant angular velocity $\vec{\omega}$ (recall, the direction of $\vec{\omega}$ is obtained by using the right-hand rule with the physical motion of points on the ring) find the magnitude and direction of the (infinitesimal) magnetic field produced at every point along the z-axis.

$$dI = \frac{dq}{T} = \frac{\sigma\, dA\, \omega}{2\pi} = \frac{\sigma\, 2\pi r\, dr\, \omega}{2\pi} = \sigma \omega r\, dr$$

$$d\vec{B} = \frac{\mu_0\, dI\, r^2}{2(r^2+z^2)^{3/2}} \hat{k} \qquad d\vec{B} = \frac{\mu_0 \sigma \omega r^3 dr}{2(r^2+z^2)^{3/2}} \hat{k}$$

or, better still...

$$\boxed{d\vec{B} = \frac{\mu_0 \sigma r^3 dr}{2(r^2+z^2)^{3/2}} \vec{\omega}}$$

- 1c) (10 points) Now lets replace the thin ring of part b with a washer that extends from $r = a$ to $r = b$. Charge is distributed over the washer with a surface charge density

$$\sigma(r) = \frac{q\,ab}{2\pi\,(b-a)} \frac{1}{r^3}$$

and it rotates with a constant angular velocity $\vec{\omega}$. Find the magnitude and direction of the magnetic field produced at every point on the z-axis.

$$d\vec{B} = \frac{\mu_0 r^3 dr\, \vec{\omega}}{2(r^2+z^2)^{3/2}} \frac{q\,ab}{2\pi(b-a)} \frac{1}{r^3}$$

$r = z\tan\theta$

$dr = \frac{z\,d\theta}{\cos^2\theta}$

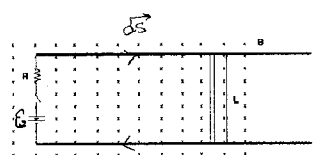
$\sqrt{r^2+z^2} = \frac{z}{\cos\theta}$

$$\vec{B} = \int d\vec{B} = \frac{\mu_0 q\,ab\,\vec{\omega}}{4\pi(b-a)} \int_a^b \frac{dr}{(r^2+z^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0 q\,ab\,\vec{\omega}}{4\pi(b-a)\,z^2} \int_{\theta_a}^{\theta_b} \cos\theta\, d\theta$$

$$\vec{B} = \frac{\mu_0 q\,ab\,\vec{\omega}}{4\pi(b-a)\,z^2} \left[\sin\theta_b - \sin\theta_a\right]$$

$$\boxed{\vec{B} = \frac{\mu_0 q\,ab\,\vec{\omega}}{4\pi\,(b-a)\,z^2} \left[\frac{b}{\sqrt{b^2+z^2}} - \frac{a}{\sqrt{a^2+z^2}}\right]}$$



2) A circuit is constructed with a battery (ξ), a resistor (R), a switch, two long, parallel, horizontal conducting rails separated by a distance L and a conducting slider (that can move without friction over the rails) of mass m and length L. The whole apparatus is completely immersed in a strong, downward, uniform magnetic field $\vec{B}$. We'll assume, for the sake of simplicity, that this external field is so strong we can safely ignore any additional magnetic field contributed by the current through the rails (in reality, you probably don't want to do that).

- 2a) (5 points) What happens when the switch is closed? Describe in as much qualitative detail as you can, the directions of the current in the circuit, the force on the slider and the resulting motion of the rail.

Current flows from the battery around the conducting loop in the clockwise direction (as drawn). A magnetic force acts on the current in the slider, pushing it to the right

- 2b) (10 points) Find the magnitude and direction of the induced EMF in the circuit and the induced current at an instant when the slider is moving with a speed v_x. Clearly explain how those directions are obtained from the mathematical calculation and explain how they are consistent with Lenz's law.

$\Phi_B = BLx$

$\mathcal{E}_i = -\frac{d\Phi_B}{dt} = -BLv_x$

$I_i = -\frac{BL}{R} v_x$ →

$\boxed{\mathcal{E}_i = -BLv_x \quad I_i = -\frac{BLv_x}{R} \quad \text{Counter-Clockwise as drawn...}}$

ok - you can drop the negative for magnitude ☺

When we calculated flux, we took $d\vec{A}$ to point along $\vec{B}$ (in). This meant the + azimuthal direction was going to be clockwise. $+v_x$ means a negative direction for $\mathcal{E}_i$, I_i (CCW). Inward flux is increasing - a CCW current counters that (to some extent) consistent with Lenz's law.

- 2c) (5 points) What is the magnitude and direction of the total current flowing in the circuit? What is the magnitude and the direction of the resulting force on the slider?

← $I_{bat} > I_i$ (How do we know that? ☺)

$I = I_{bat} + I_i$

$I = \frac{\mathcal{E}}{R} - \frac{BLv_x}{R}$

$\vec{F}_B = I\vec{L} \times \vec{B}$

$F_B = \left(\frac{\mathcal{E}}{R} - \frac{BLv_x}{R}\right) BL \sin(90^\circ)$

$F_B = \frac{(\mathcal{E} - BLv_x)BL}{R}$

$$\boxed{I = \frac{\mathcal{E} - BLv_x}{R} \;\text{(Clockwise)} \qquad F_B = \frac{(\mathcal{E} - BLv_x)BL}{R} \;\text{(to the right)}}$$

- 2d) (10 points) Assuming the slider starts at rest the instant the switch is closed ($t = 0$), find the velocity of the slider at every instant after the switch is closed. Explain by first principles why you know the slider will reach a terminal velocity. What is the speed of the slider at terminal velocity?

$\Sigma F_x = ma_x$

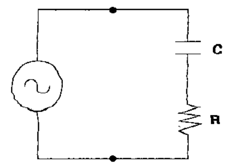
$(\mathcal{E} - BLv_x)\frac{BL}{R} = m\frac{dv_x}{dt}$

$\left(\frac{\mathcal{E}}{BL} - v_x\right)\frac{B^2L^2}{R} = m\frac{dv_x}{dt}$

$\int_0^{v_x} \frac{dv_x}{v_x - \mathcal{E}/BL} = -\int_0^t \frac{dt}{mR/B^2L^2}$

$$\boxed{v_x = \frac{\mathcal{E}}{BL}\left(1 - e^{\frac{-t}{mR/B^2L^2}}\right) \qquad v_{term} = \frac{\mathcal{E}}{BL}}$$

As the slider accelerates, the increase in v_x increases I_i to the point where it balances I_{bat}. With no net current, the force vanishes - the slider can no longer accelerate - it's stuck at v_{term}...



3) An capacitor (C) and a resistor (R) are connected in series across a source of alternating EMF ($\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$). You may do the following calculations in complex space, but keep in mind, the final answers must all be real.

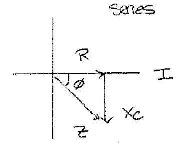
- 3a) (10 points) What is the impedance of RC combination? What is the amplitude of the current that passes through it? Does the current through this impedance lead or lag the voltage across it? By how much?

$$\boxed{Z = \sqrt{(1/\omega C)^2 + R^2}}$$

$$I_{max} = \frac{\mathcal{E}_{max}}{\sqrt{(1/\omega C)^2 + R^2}}$$

Current leads the Voltage by $\phi = \tan^{-1} \frac{1}{\omega C R}$

Series



$Z = \sqrt{R^2 + X_C^2}$

$X_C = 1/\omega C$

$I_{max} = \mathcal{E}_{max}/Z$

$\phi = \tan^{-1} \frac{X_C}{R}$

- 3b) (10 points) What is the voltage amplitude across the resistor? What is the voltage amplitude across the capacitor? Will the sum of these voltage amplitudes equal the voltage amplitude of the source? Explain.

$\Delta V_{R,max} = I_{max} R$

$\Delta V_{C,max} = I_{max} X_C$

$\Rightarrow$

$$\Delta V_{R,max} = \frac{\mathcal{E}_{max} R}{\sqrt{(1/\omega C)^2 + R^2}}$$

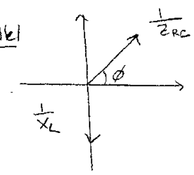
$$\Delta V_{C,max} = \frac{\mathcal{E}_{max}/\omega C}{\sqrt{(1/\omega C)^2 + R^2}}$$

$\Delta V_{R max} + \Delta V_{C max} \neq \mathcal{E}_{max}$ (They're not in phase- they don't reach their maximums at the same time)

- 3c) (10 points) For a lot of good reasons, it is often desirable to present the source with a purely resistive load. We can tune-out the reactance in the RC network by adding an inductor in parallel with it (across the two dots in the circuit). What value should the inductor have? What will be the value of the new impedance seen by the source? Will current through this new LRC combination lead or lag the voltage across it? By how much?

Fun! Study the impedance diagram in part a before you start...

Parallel



We want...

$\frac{1}{X_L} = \frac{1}{Z_{RC}} \sin\phi$ ← but $\sin\phi = \frac{X_C}{Z_{RC}}$ (from part a)

$\frac{1}{X_L} = \frac{X_C}{Z_{RC}^2}$

$\frac{1}{\omega L} = \frac{1}{\omega C} \frac{1}{(R^2 + (1/\omega C)^2)}$

$$L = C(R^2 + (1/\omega C)^2)$$

$\frac{1}{Z} = \frac{1}{Z_{RC}} \cos\phi = \frac{1}{Z_{RC}} \frac{R}{Z_{RC}}$

$Z = \frac{Z_{RC}^2}{R}$

$L = C(R^2 + (1/\omega C)^2)$

$Z = \frac{1}{R}(R^2 + (1/\omega C)^2)$

The load is resistive - Current will be in phase with the Voltage!

← You are, of course, welcome to do this with complex impedances, if you know what you're doing

$\frac{1}{Z} = \frac{1}{iX_L} + \frac{1}{R - iX_C}$

Same Result "