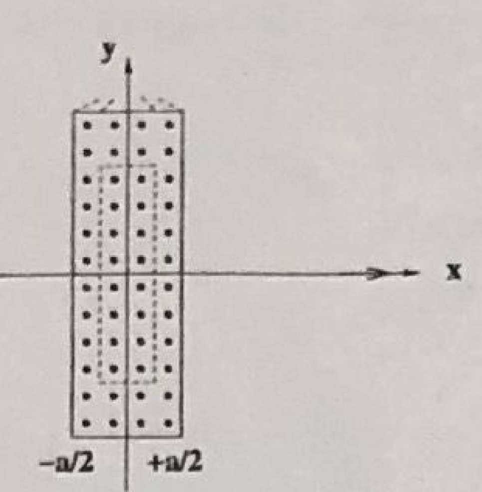


A non-uniform current described by the current density

$$\vec{J} = \vec{J}_0 \cos\left(\pi \frac{x}{a}\right)$$

flows out of the plane of the page through a conducting sheet that extends horizontally from $x = -\frac{a}{2}$ to $x = +\frac{a}{2}$ and vertically from $y = -\infty$ to $y = +\infty$.

$$I = \int \vec{J} \cdot d\vec{A} \qquad d\vec{A} = 2y\,dx\,\hat{z}$$

- 1a) (10 pts) How much current will pass through a rectangular loop that extends horizontally from $-x$ to $+x$ and vertically from $-y$ to $+y$ if i) $x < \frac{a}{2}$ and ii) $x > \frac{a}{2}$

$(x \le \frac{a}{2})$ $\quad I = \int_{-x}^{x} J_0 \cos\left(\frac{\pi x}{a}\right) 2y\,dx$

$I = \frac{2ay}{\pi} J_0 \sin\left(\frac{\pi x}{a}\right)\Big|_{-x}^{x}$

$(x > \frac{a}{2})$ $\quad I = \int_{-a/2}^{a/2} J_0 \cos\left(\frac{\pi x}{a}\right) 2y\,dx$

$I = \frac{2ay}{\pi} J_0 \sin\left(\frac{\pi x}{a}\right)\Big|_{-a/2}^{a/2}$

$$\boxed{I = \begin{cases} \frac{4ay}{\pi} J_0 \sin\left(\frac{\pi x}{a}\right) & (x \le a/2) \\ \frac{4ay}{\pi} J_0 & (x > a/2) \end{cases}}$$

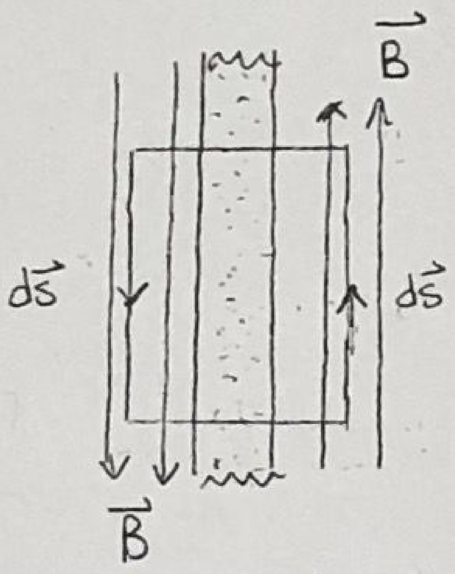
- 1b) (10 pts) Find the (vector) magnetic field for *all* points on the x-axis.

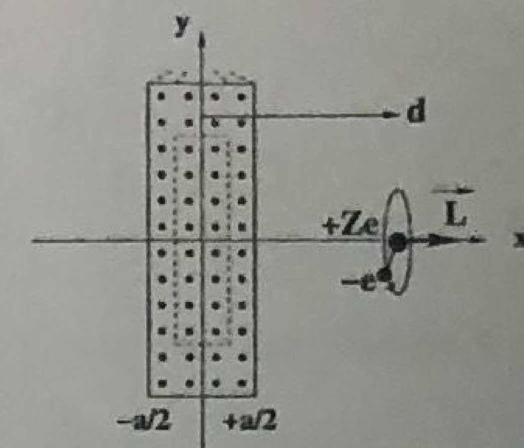
Ampere's Law: $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$; use loop from (a).

$(x \le a/2)$ $\quad B2y + B2y = \frac{\mu_0 4ay}{\pi} J_0 \sin\left(\frac{\pi x}{a}\right)$

$(x > a/2)$ $\quad B2y + B2y = \frac{\mu_0 4ay}{\pi} J_0$

$$\boxed{\vec{B} = \begin{cases} \frac{\mu_0 a}{\pi} J_0 \sin\left(\frac{\pi x}{a}\right) \hat{j} & (|x| \le a/2) \\ \frac{\mu_0 a}{\pi} J_0 \frac{x}{|x|} \hat{j} & (|x| > a/2) \end{cases}}$$





- 1c) (10 pts) An electron (mass m, charge $-e$) orbits a nucleus (charge $+Ze$) located at the point $< d, 0, 0 >$. If the electron has an orbital angular momentum given by $\vec{L} = L\hat{x}$, find the magnitude and direction of i) the magnetic dipole moment associated with the electron's orbital motion and ii) the torque on the atom due to the magnetic field created by the current sheet.

$\vec{L} = I\vec{\omega}$

$L = mr^2\omega$

$L = \frac{2\pi m r^2}{T}$

$T = \frac{2\pi m r^2}{L}$

$I = \frac{e}{T}$

$I = \frac{eL}{2\pi m r^2}$

$\mu = NIA$

$\mu = \frac{eL}{2\pi m r^2} \cdot \pi r^2$

$\mu = \frac{e}{2m} L$

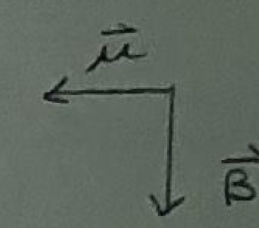
Important because the e^- is negative, current flows opposite its physical motion. This means $\vec{\mu}$ points opposite $\vec{L}$.

$$\boxed{\vec{\mu} = \frac{-e}{2m} \vec{L}}$$

$\vec{\tau} = \vec{\mu} \times \vec{B}$

$\vec{\tau} = \mu B \hat{k}$

$\vec{\tau} = \frac{e}{2m} \frac{\mu_0 a}{\pi} J_0 \hat{k}$



$$\boxed{\vec{\tau} = \frac{\mu_0 a e}{2\pi m} J_0 \cancel{\hat{k}} \; -\hat{z}}$$

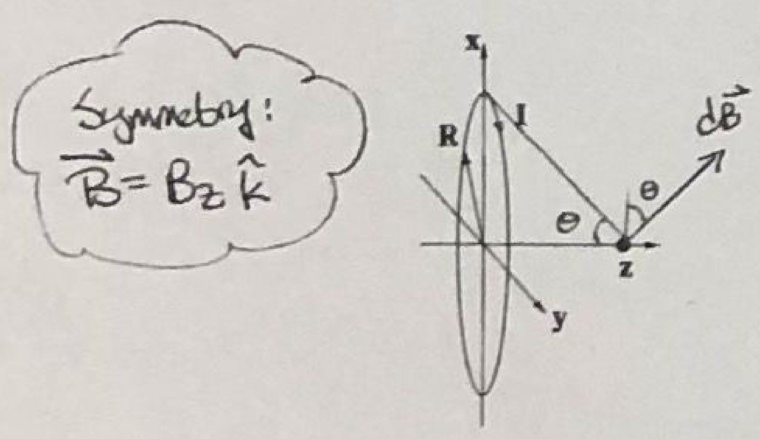


Figure 1

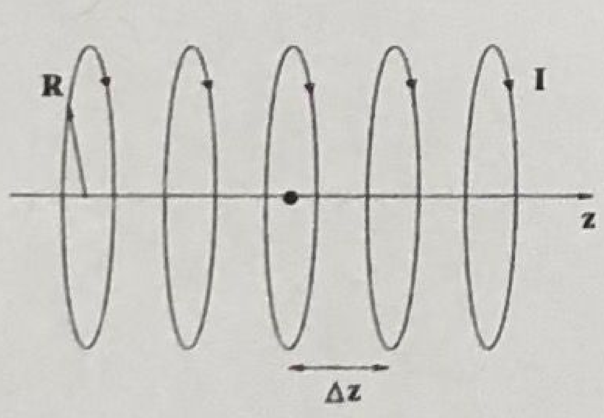


Figure 2

- 2a) (10 points) Figure 1 shows a circular conducting loop of radius R, situated in the x, y plane, centered on the origin. It carries a current I which is directed in the $+y$ direction at the top of the loop. Derive the magnetic field (magnitude and direction) for all points on the z-axis.

$dB = \frac{\mu_0 I}{4\pi} \frac{ds\,(1)\sin 90°}{R^2 + z^2}$

$dB_z = dB \sin\theta = dB \frac{R}{\sqrt{R^2+z^2}}$

$dB_z = \frac{\mu_0 I R}{4\pi (R^2+z^2)^{3/2}} ds$

$B_z = \int dB_z = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$

$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \hat{k}}$$

- 2b) (5 points) In Figure 2, we find a collection of conducting loops (Radius R, oriented parallel to the x, y-plane, centered on the z-axis) that extend in steps of Δz from $-N\Delta z$ to $+N\Delta z$. Each carries a current I which is directed in the $+y$ direction at the top of the loop. Derive the magnetic field (magnitude and direction) at the center of the assembly.

Superposition

$$\boxed{\vec{B} = \sum_{i=-N}^{N} \frac{\mu_0 I R^2}{2(R^2 + (i\Delta z)^2)^{3/2}} \hat{k}}$$

Note that the field due to loops centered on the negative side of z, as well as the field due to loops on the positive side all point in the $+z$ direction

- 2c) (5 points) Use $2N\Delta z = L$ and $z = (index) * \Delta z$ to re-write your answer to the previous part in a form suggestive of the Riemann sum. (Hint: there are many ways to write '1'.)

$\vec{B} = \frac{\mu_0 I R^2}{2} \sum_{i=-N}^{N} \frac{1}{(R^2 + (i\Delta z)^2)^{3/2}} \hat{k}$ $\quad \leftarrow \quad 1 \equiv \frac{2N}{L} \Delta z$

$N = \frac{L}{2\Delta z}$

$z = i\Delta z$

$\vec{B} = \frac{\mu_0 I R^2}{2} \cdot \frac{2N}{L} \lim_{\Delta z \to 0} \sum_{i\Delta z = -L/2}^{L/2} \frac{\Delta z}{(R^2+z^2)^{3/2}} \hat{k}$

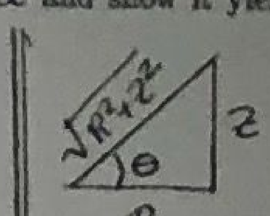
$$\boxed{\vec{B} = \mu_0 \frac{N}{L} I R^2 \lim_{\Delta z \to 0} \sum_{i\Delta z = -L/2}^{L/2} \frac{\Delta z}{(R^2+z^2)^{3/2}} \hat{k}}$$

- 2c) (10 points) Convert the Riemann sum to an integral and evaluate $\vec{B}$ at the center of this solenoid of finite length. Evaluate your answer (carefully) in the limit that $L \to \infty$ and show it yields the correct result.

$\vec{B} = \mu_0 \frac{N}{L} I R^2 \int_{-L/2}^{L/2} \frac{dz}{(R^2+z^2)^{3/2}} \hat{k}$

$\vec{B} = \frac{\mu_0 N I R^2}{L} \frac{R}{R^3} \int_{\theta_-}^{\theta_+} \frac{\cos^3\theta}{\cos^2\theta} d\theta \; \hat{z}$

$\vec{B} = \frac{\mu_0 N I}{L} \sin\theta \Big|_{\theta_-}^{\theta_+} \hat{z}$



$z = R\tan\theta$

$dz = \frac{R\,d\theta}{\cos^2\theta}$

$\sqrt{R^2+z^2} = \frac{R}{\cos\theta}$

z runs from $-L/2$ to $L/2$

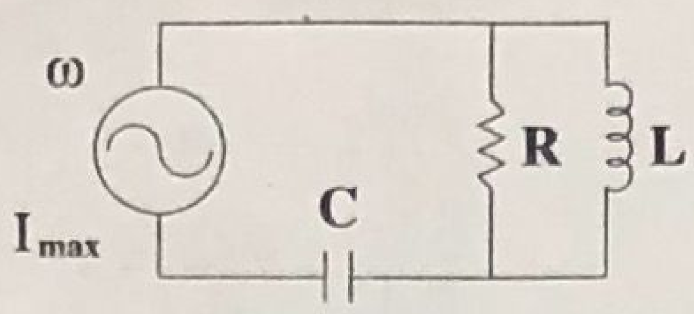
$\sin\theta_+ = \frac{L/2}{\sqrt{R^2+(L/2)^2}}$

$\sin\theta_- = -\sin\theta_+$

$$\boxed{\vec{B} = \mu_0 \left(\frac{2N}{L}\right) I \frac{L/2}{\sqrt{R^2+(L/2)^2}} \hat{k}}$$

$\lim_{L\to\infty}$: $\vec{B} = \mu_0 \left(\frac{2N}{L}\right) I \hat{k}$ $\quad \leftarrow \quad \frac{2N}{L} = \frac{1}{\Delta z} = n \leftarrow$ turns per unit length

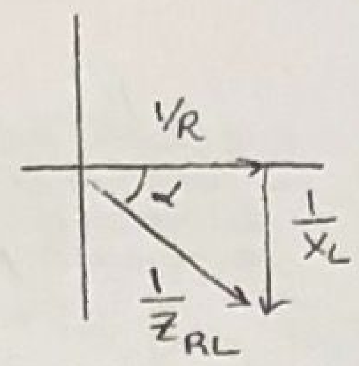
$\vec{B} = \mu_0 n I \hat{k}$ ✓✓



3) A sinusoidally-varying **current** of amplitude I_{max} and angular frequency ω drives the RLC network shown above.

- 3a) (5 pts) Find the impedance of the RL combination.

$$\frac{1}{Z_{RL}} = \sqrt{\frac{1}{X_L^2} + \frac{1}{R^2}}$$



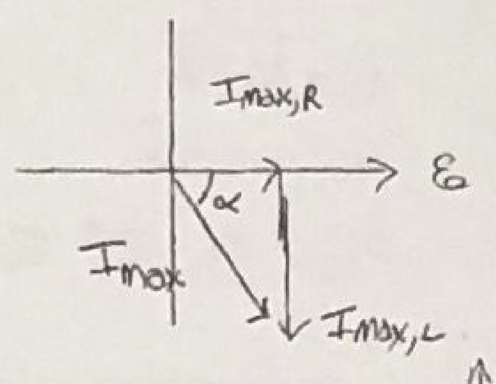
$$\boxed{Z_{RL} = \frac{1}{\sqrt{\left(\frac{1}{\omega L}\right)^2 + \left(\frac{1}{R}\right)^2}}}$$

- 3b) (5 pts) Will the voltage across the RL combination lead or lag the current that passes through it? By how much?

$$\tan\alpha = \frac{R}{X_L}$$

The Current will lag the Voltage by

$$\alpha = \tan^{-1}\frac{R}{\omega L}$$



parallel! Voltage in Common

- 3c) (10 pts) What is the maximum current that will flow through the inductor? What is the maximum current that will flow through the resistor?

$$\Delta V_{L,max} = I_{max,L} X_L$$

$$I_{max} Z_{LR} = I_{max,L} X_L$$

$$I_{max,L} = I_{max} \frac{Z_{LR}}{X_L}$$

$$\boxed{I_{max,L} = I_{max} \frac{1}{\omega L \sqrt{\left(\frac{1}{\omega L}\right)^2 + 1/R^2}}}$$

$$\Delta V_{R,max} = I_{max,R} R$$

$$I_{max} Z_{LR} = I_{max,R} R$$

$$I_{max,R} = I_{max} \frac{Z_{LR}}{R}$$

$$\boxed{I_{max,R} = \frac{1}{R\sqrt{\left(\frac{1}{\omega L}\right)^2 + 1/R^2}}}$$

- 3d) (5 pts) Will $I_{C,max} = I_{L,max} + I_{R,max}$? Explain.

No... the currents through the inductor and resistor are not in phase

- 3e) (5 pts) Draw a phasor diagram (in impedance space) to show how you would combine the capacitor with the RL combination to find the total impedance seen by the source. Label the contributions from the capacitor and the RL combination clearly, and make sure you clearly identify the relevant angles for anything that doesn't sit on an axis.