MT1 Physics 1C(1), [redacted] F17

Full Name (Printed) [redacted]

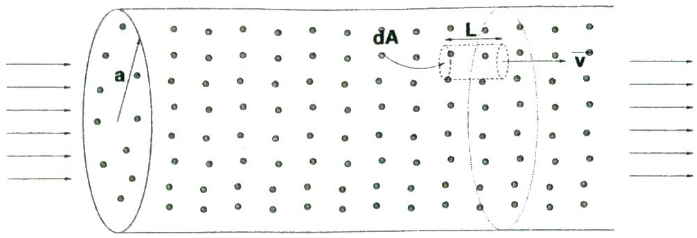
Full Name (Signature) [redacted]

Student ID Number [redacted]

Seat Number D9

Problem	Grade
1	28 /30
2	30 /30
3	09 /30
Total	67 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**



$z = \int \rho \, dV$

$I = \int J \, dA$

1) A cylindrical particle beam of radius a is made up of electrically-charged particles traveling with a speed v (along the direction of the longitudinal axis). The volume charge density of particles within the beam is given by $\rho(r) = \rho_0 \left(\frac{r}{a}\right)^n$. ($r$ is measured in cylindrical coordinates and n is a positive integer)

- 1a) (5 points) Consider the little sub-volume shown in the diagram. In terms of the quantities you've been given, how much charge is contained within that sub-volume? How long will it take for the charge in the sub-volume to pass through the reference plane? How much will the sub-volume contribute to the current flowing in the beam? From your answers, determine the current density associated with the beam.

$\boxed{t = \frac{L}{v}}$ $\quad dq = \int \rho \, dV = \int \rho_0 \left(\frac{r}{a}\right)^n L \, dA$; $\boxed{dq = \rho_0 L \int \left(\frac{r}{a}\right)^n dA}$

$$dI = \frac{dq}{t} = \frac{\rho_0 L \int \left(\frac{r}{a}\right)^n dA}{L/v} = \boxed{\rho_0 v \int \left(\frac{r}{a}\right)^n dA}$$

$$J = \frac{dI}{dt} = \boxed{\rho_0 v \left(\frac{r}{a}\right)^n}$$

5

- 1b) (10 points) How much current passes through a circular loop of radius r if that loop is oriented perpendicular to the beam and is centered on its longitudinal axis? Solve for both $r < a$ and $r > a$. (If you didn't solve the first part, use $\vec{J} = \vec{J_0} \left(\frac{r}{a}\right)^\alpha$).

$(r < a)$

$$I = \int J \, dA = \int_0^r \rho_0 v \left(\frac{r}{a}\right)^n 2\pi r \, dr$$

$$= \frac{2\pi \rho_0 v}{a^n} \int_0^r r^{n+1} dr = \frac{2\pi \rho_0 v}{a^n} \left(\frac{r^{n+2}}{n+2}\right)\Bigg|_0^r$$

$$\boxed{I = \frac{2\pi \rho_0 v r^{n+2}}{(n+2) a^n}}$$

$(r > a)$

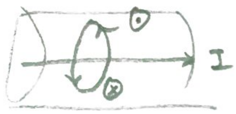
$$I = \int J \, dA = \int_0^a \rho_0 v \left(\frac{r}{a}\right)^n 2\pi r \, dr + 0$$

$$= \frac{2\pi \rho_0 v (a)^{n+2}}{(n+2) a^n} = \frac{2\pi \rho_0 v a^2}{n+2}$$

$$\boxed{I = \frac{2\pi \rho_0 v a^2}{n+2}}$$

- 1c) (10 points) Find the magnitude and direction of the magnetic field at points inside and outside the beam, assuming the charge carriers are positive.

$(r < a)$



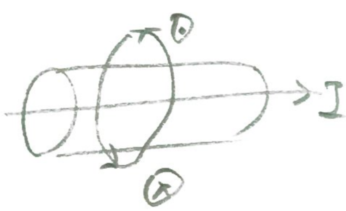
$$\oint \vec{B}(r) \cdot d\vec{s} = \mu_0 I_{enc}$$

$$B(r)\, 2\pi r = \mu_0 I_{enc}$$

$$B(r) = \frac{\mu_0 I_{enc}}{2\pi r} = \frac{\mu_0}{2\pi r}\left(\frac{2\pi \rho_0 v r^{n+2}}{(n+2)a^n}\right)$$

$$\boxed{B(r) = \frac{\mu_0 \rho_0 v r^{n+1}}{(n+2)a^n}}$$

$(r > a)$



$$\oint \vec{B}(r) \cdot d\vec{s} = \mu_0 I_{en}$$

$$B(r)\, 2\pi r = \mu_0 I_{enc}$$

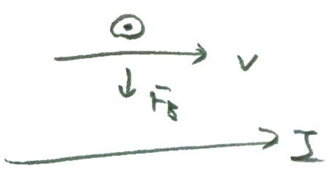
$$B(r) = \frac{\mu_0}{2\pi r}\left(\frac{2\pi \rho_0 v a^2}{n+2}\right)$$

$$\boxed{B(r) = \frac{\mu_0 \rho_0 v a^2}{r(n+2)}}$$

10

- 1d) (5 points) What force will a particle of charge $+q$ experience if it is located a distance r from the longitudinal axis of the beam? What implications does this have for the beam?

$$\vec{F}_B = q\vec{v} \times \vec{B}$$



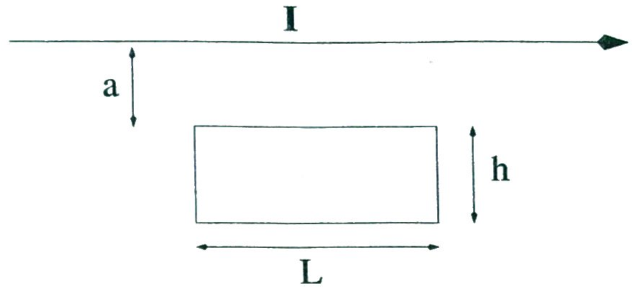
$$\vec{F} = (+q)\left(\vec{v} \times \left(\frac{\mu_0 \rho_0 v r^{n+1}}{(n+2)a^n}\hat{\phi}\right)\right)$$

$$= q|v|\left|\frac{\mu_0 \rho_0 v r^{n+1}}{a^n(n+2)}\right| \sin(90°)$$

3

$$\boxed{F = \frac{q v \mu_0 \rho_0 v r^{n+1}}{(n+2)a^n}}$$

The beam will get smaller and smaller



2) A current $I(t) = I_{max}\sin(\omega t + \phi)$ flows through a long, thin wire that runs parallel to (and coplanar with) a rectangular conducting loop of length L and height h, a distance a from the loop, as shown.

- 2a) (15 points) What is the magnitude of the induced electromotive force around the loop?

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

$$B(r)\, 2\pi r = \mu_0 I_{enc}$$

$$B(r) = \frac{\mu_0 I_{enc}}{2\pi r}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\int \vec{B}(r)\cdot d\vec{A}\right) = -\frac{d}{dt}\left(\int B(r)\, dA\right)$$

$$\int B(r)\, dA = \int_a^{a+h}\int_0^L \left(\frac{\mu_0 I_{enc}}{2\pi r}\right) dx\, dr = \int_a^{a+h}\left(\frac{\mu_0 L I_{enc}}{2\pi r}\right) dr$$

$$= \frac{\mu_0 L}{2\pi} I_{enc} \ln|r| \Big|_a^{a+h} = \frac{\mu_0 L I_{enc}}{2\pi} \ln\left|\frac{a+h}{a}\right|$$

$$\mathcal{E} = -\frac{d}{dt}\left(\frac{\mu_0 L}{2\pi}\ln\left|\frac{a+h}{a}\right| \cdot I_{max}\sin(\omega t+\phi)\right)$$

$$= -\left(\frac{\mu_0 L}{2\pi}\ln\left|\frac{a+h}{a}\right| I_{max}\,\omega\cos(\omega t+\phi)\right)$$

$$\boxed{\mathcal{E} = -\left(\frac{\mu_0}{2\pi} L\omega \ln\left|\frac{a+h}{a}\right|\right) I_{max}\cos(\omega t+\phi)}$$

+15

$\phi = L I$ $\qquad \mathcal{E} = -\frac{d\phi}{dt} = -L\frac{dI}{dt}$

- 2b) (10 points) What is the mutual inductance between the wire and the loop?

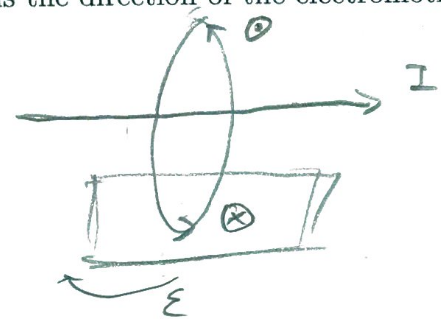
$$L = \left|\frac{\mathcal{E}}{dI/dt}\right| = \left|\frac{-\frac{\mu_0}{2\pi} L\omega \ln\left|\frac{a+h}{a}\right| I_{max}\cos(\omega t+\phi)}{\omega I_{max}\cos(\omega t+\phi)}\right|$$

$$\boxed{M \cancel{L} = \frac{\mu_0}{2\pi} L \ln\left|\frac{a+h}{a}\right|}$$

inductance $\qquad$ length

+10

- 2c) (5 points) If the current is flowing in the direction shown, and decreasing in magnitude, what is the direction of the electromotive force induced in the wire loop?

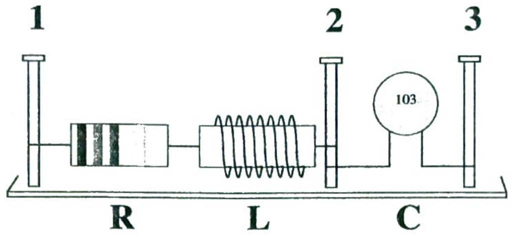


If $\frac{dI}{dt} < 0$,

to counteract the decrease in flux,

$\boxed{\mathcal{E} \text{ is clockwise}}$ ✓

+5



$X_c = \frac{1}{\omega C}$, $X_L = \omega L$

3) Shopping around on EBay, you've managed to find and buy a large box full of small circuit boards for next to nothing. When it arrives, you notice that each board has a resistance R, an inductance L and a capacitance C mounted between conductive terminals labeled **1**, **2** and **3** as shown above. The values of R and C are clearly marked, the inductor, unfortunately, is not labeled with a value.

The boards are marked with the name of a popular speaker manufacturer, leading you to believe the circuits are used as cross-over networks, designed to send high-frequency sound preferentially to the high-frequency speaker (the "tweeter") and low-frequency sound preferentially to the low-frequency speaker (the "woofer").

4.

- 3a) (5 points) Between which pair of terminals should you connect the speaker wires that run from your stereo? Between which pair of terminals should you wire the tweeter? Between which pair of terminals should you wire the woofer? **Explain...**

$X_c = \frac{1}{\omega C}$, $X_L = \omega L$

stereo: 1, 3

tweeter: 1, 2

woofer: 2, 3

explain?

5.

- 3b) (5 points) Suppose you hook a signal generator in series with an ammeter across the terminals labeled **1** and **3** (the speakers are not attached). As you vary the frequency of the generator, you note that you get maximum current draw from the generator when the output signal is set to an angular frequency Ω. What is the value of the inductor?

$$\omega = \frac{1}{\sqrt{LC}}$$

$$LC = \frac{1}{\omega^2}$$

$$\boxed{L = \frac{1}{C\omega^2}}$$

0 • 3c) (5 points) The signal generator has an rms output voltage of ξ_{rms}. If we leave the generator set to the angular frequency Ω (of part b), what will be the rms voltage measured between terminals **1** and **2** ($\xi_{12}(\Omega)$)? ...between terminals **2** and **3** ($\xi_{23}(\Omega)$)?

6 • 3d) (5 points) Compare $\xi_{12}(\Omega) + \xi_{23}(\Omega)$ to ξ_{rms}. Explain.

0 • 3e) (10 points) Find the angular frequency at which $\xi_{12} = \xi_{23}$. What is the significance of this frequency?