MT1 Physics 1C(2), F13

Full Name (Printed) ______________________

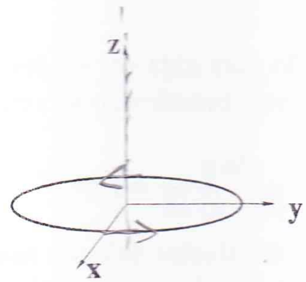
Full Name (Signature) ______________________

Student ID Number ______________________

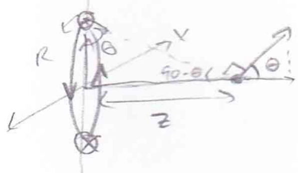
Seat Number ______________________

Problem	Grade
1	23 /30
2	21 /30
3	21 ~~19~~ 21 /30
Total	65 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**



- 1a) (10 points) A circular, conducting loop of radius R lies in the x,y-plane, centered on the origin. A current I flows through the loop such that at $x = +R$, the current is headed in the $+\hat{y}$ direction, and at $x = -R$ the current is headed in the $-\hat{y}$ direction. Derive the resultant magnetic field (magnitude and direction) at every point along the z-axis.



z-axis, only z-component exists,

$$dB_z = dB\cos\theta$$

$$\int dB_z = \frac{\mu_0 I}{4\pi}\int \frac{dl \times \hat{r}}{r^2}\cos\theta$$

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{dl\times\hat{r}}{z^2+R^2}\cos\theta$$

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{1}{z^2+R^2}\cdot\frac{R}{\sqrt{z^2+R^2}}\,dl$$

$$B_z = \frac{\mu_0 I R}{4\pi (z^2+R^2)^{3/2}}\cdot 2\pi R = \cancel{\mu_0}$$

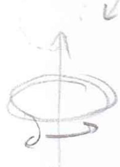
$$\boxed{\vec{B}(z) = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}\hat{z}}$$

9

- 1b) (10 points) Let's replace the loop with an infinitesimally-thin, uniform ring of electric charge that extends from r to $r + dr$. If the surface charge-density on the ring is given by σ and the ring rotates about the z-axis with a constant angular velocity $\vec{\omega}$ (recall, the direction of $\hat{\omega}$ is obtained by using the right-hand rule with the physical motion of points on the ring) find the magnitude and direction of the (infinitesimal) magnetic field produced at every point along the z-axis.

7

$A \approx 2\pi r$



$$dq = \sigma\, dA$$
$$\sigma(2\pi r\, dr)$$

$$A = \pi(r+dr)^2 - \pi r^2 \qquad 2r\,dr$$
$$A = \pi r^2 + 2\pi r\,dr + \pi dr^2 - \pi r^2$$
$$A = 2\pi r\,dr + \pi dr^2\ ?$$

$$I = \frac{dq}{T} \qquad T = 1/f = \qquad \omega = 2\pi f$$
$$T = \frac{2\pi}{\omega}$$

$$dI = \frac{2\pi\, dq}{\omega}$$

$$I = \int \frac{2\pi\, dq}{\omega}$$

$$I = \frac{2\pi}{\omega}\int dq = \frac{2\pi}{\omega}\int_r^{R} 2\pi\sigma\, dr = \frac{4\pi^2\sigma}{\omega} r$$

$$\vec{B}(z) = \frac{\mu_0\left(\frac{2\pi}{\omega}\right)\int dq\, R}{2(z^2+R^2)^{3/2}}\hat{z}$$

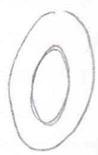
$$\boxed{\vec{B}(z) = \frac{\mu_0 \frac{4\pi^2\sigma}{\omega} R}{2(z^2+R^2)^{3/2}}\hat{z}}$$

- 1c) (10 points) Now lets replace the thin ring of part b with a washer that extends from $r = a$ to $r = b$. Charge is distributed over the washer with a surface charge density

$$\sigma(r) = \frac{q\,ab}{2\pi\,(b-a)}\,\frac{1}{r^3}$$

and it rotates with a constant angular velocity $\vec{\omega}$. Find the magnitude and direction of the magnetic field produced at every point on the z-axis.

7



$dq = \sigma\, dA$ $\quad A = \pi r_2^2 - \pi r_1^2$

$dq_{1\text{-ring}} = \sigma\, dA_{1\text{-ring}}$ $\quad dA = \pi$

 ← treat as rings

$q_{1\text{-ring}} = \sigma$

$I = \frac{dq}{T}$ $\quad dI = \frac{2\pi\, dq}{\omega}$ $\quad$ Find dq interms of dr

$dq = \sigma\, dA$

$dA = ?\, dr?$

Current is just charge over period here.

$I = \frac{2\pi}{\omega}\int dq = \frac{2\pi}{\omega}$

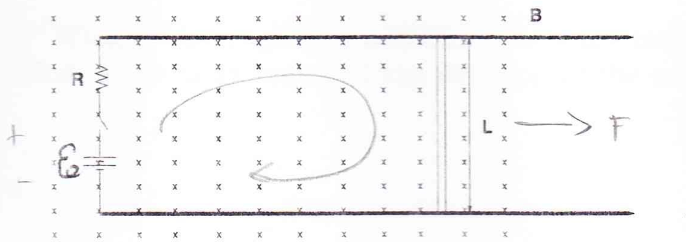
$I = \frac{2\pi}{\omega}\int \sigma(r)\, dA$

$$\boxed{B(z) = \int \frac{\mu_0 \frac{2\pi}{\omega}\int \sigma(r)\, dA}{2\left(z^2 + R^2\right)^{3/2}}\,\hat{z}}$$

?

$$\int_a^b \sigma(r)\, dr = \frac{q\,ab}{2\pi(b-a)}\left[\frac{1}{4} r^4\right]_a^b$$

$$= \frac{q\,ab}{2\pi(b-a)}\,\frac{1}{4}\left(b^4 - a^4\right)$$



2) A circuit is constructed with a battery (ξ), a resistor (R), a switch, two long, parallel, horizontal conducting rails separated by a distance L and a conducting slider (that can move without friction over the rails) of mass m and length L. The whole apparatus is completely immersed in a strong, downward, uniform magnetic field $\vec{B}$. We'll assume, for the sake of simplicity, that this external field is so strong we can safely ignore any additional magnetic field contributed by the current through the rails (in reality, you probably don't want to do that).

- 2a) (5 points) What happens when the switch is closed? Describe in as much qualitative detail as you can, the directions of the current in the circuit, the force on the slider and the resulting motion of the rail.

4

Once the switch is closed, current will begin to flow clockwise, → $i\ B \times \ell$
i.e, down the bar. Since it is able to move, $F = BiL$, so the B
directed ~~out of the page~~ into the page, the length of the bar (current direction is downwards,
so the force will be to the right. ~~S~~ The bar moves, so the flux through
the loop changes, inducing a counter current, slowing the bar down (to left).
This counter-force isn't enough to stop ~~it~~ the bar's motion, however.

2 mistakes cancel..

- 2b) (10 points) Find the magnitude and direction of the induced EMF in the circuit and the induced current at an instant when the slider is moving with a speed v_x. Clearly explain how those directions are obtained from the mathematical calculation and explain how they are consistent with Lenz's law.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\Phi_B \frac{d}{dt} = -\frac{d}{dt}\int B \cdot dA$$

1

$$= -\frac{d}{dt}\int B \cdot \ell \cdot dx = -BL$$

Faraday's Law says that change in flux in time is the induced EMF. Flux is just $\int B \cdot dA$. Since B is constant, A is the rectangle of the circ

$$\mathcal{E}(v) = -BLv$$

$$\boxed{\mathcal{E}(v_x) = -BLv_x}$$

1

$$\boxed{I = \mathcal{E}/R = \frac{-BLv_x}{R}}$$

2

huh?

If we take the positive EMF as from top to bottom, we see that these are consistent.

Direction dA

The induced emf is negative, providing a current counterclockwise that slows the bar down.

1

We integrate over the distance along the rail. The change in x over time is simply velocity. Current is just the voltage by resistance.

- 2c) (5 points) What is the magnitude and direction of the total current flowing in the circuit? What is the magnitude and the direction of the resulting force on the slider?

$$I_{source} = \mathcal{E}/R$$

$$I_{induced} = -\frac{BLv_x}{R}$$

$$I_{total} = I_{source} + I_{induced} = \frac{\mathcal{E}}{R} - \frac{BLv_x}{R} \quad 2$$

$$\boxed{I_{total} = \frac{\mathcal{E} - BLv_x}{R}}$$

this quantity is always going to be positive, or zero because at some terminal speed v_t the I will cause canceling forces and the bar will stop accelerating. (Direction is ~~↓~~ ↓) or +.

2

Direction?

- 2d) (10 points) Assuming the slider starts at rest the instant the switch is closed ($t = 0$), find the velocity of the slider at every instant after the switch is closed. Explain by first principles why you know the slider will reach a terminal velocity. What is the speed of the slider at terminal velocity?

$$F_B = BiL$$

$$= B\left(\frac{\mathcal{E} - BLv}{R}\right)L$$

$$0 = \frac{\mathcal{E}BL - B^2L^2v}{R}$$

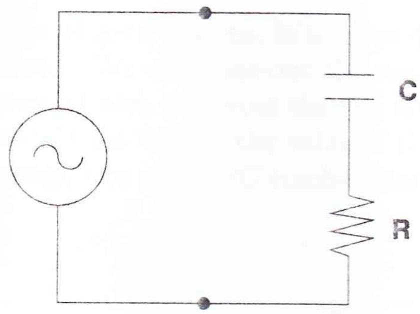
$$\mathcal{E}BL = B^2L^2v$$

$$\boxed{v_{term} = \frac{\mathcal{E}}{BL}} \quad 4$$

$\mathcal{E}BL$

~~At~~ There will only be acceleration if $F_B \neq 0$. It is defined by BiL. Since B and L are constant, we care about when $i = 0$. This occurs when the battery current and the induced current cancel, which does indeed occur. There must be a terminal velocity then. 4

F→C: → ...



3) An capacitor (C) and a resistor (R) are connected in series across a source of alternating EMF ($\xi(t) = \xi_{max}\cos(\omega t)$). You may do the following calculations in complex space, but keep in mind, the final answers must all be real.

ICE
legs

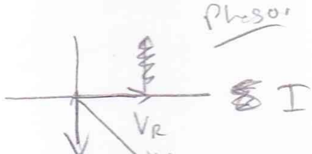
- 3a) (10 points) What is the impedance of RC combination? What is the amplitude of the current that passes through it? Does the current through this impedance lead or lag the voltage across it? By how much?

phasor

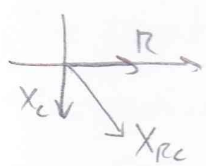
$X_c = 1/\omega C$

$V = IR$

$\frac{V}{I} = R$



divide by I for resistance



$\tan\phi = \frac{X_c}{R}$

$$\boxed{X_{RC} = \sqrt{R^2 + X_c^2} = \sqrt{R^2 + (1/\omega C)^2}}$$

$$I = V/R = \frac{\mathcal{E}_{max}\cos(\omega t)}{\sqrt{R^2+(1/\omega C)^2}}$$

$$\boxed{I_{max} = \mathcal{E}_{max} / \sqrt{R^2+(1/\omega C)^2}}$$

It ~~lags~~, by

$$\boxed{\arctan\left(\frac{X_c}{R}\right)}$$

−1

- 3b) (10 points) What is the voltage amplitude across the resistor? What is the voltage amplitude across the capacitor? Will the sum of these voltage amplitudes equal the voltage amplitude of the source? Explain.

$V_R = IR$

$$V_R = \frac{\mathcal{E}_{max} R\cos(\omega t)}{\sqrt{R^2+(1/\omega C)^2}}$$

$$V_{R,max} = \frac{\mathcal{E}_{max} R}{\sqrt{R^2+(1/\omega C)^2}}$$

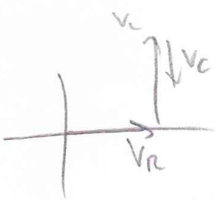
$V_c = I X_c$

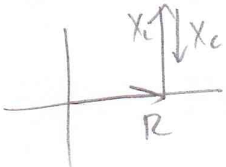
$$V_{c,max} = \frac{\mathcal{E}_{max}}{\omega C\sqrt{R^2+(1/\omega C)^2}}$$

No. Since the voltage across the capacitor lags by some ϕ, the summation of voltages will not provide the total $\mathcal{E}(t)$ of the circuit.

✓

- 3c) (10 points) For a lot of good reasons, it is often desirable to present the source with a purely resistive load. We can tune-out the reactance in the RC network by adding an inductor in parallel with it (across the two dots in the circuit). What value should the inductor have? What will be the value of the new impedance seen by the source? Will current through this new LRC combination lead or lag the voltage across it? By how much?





$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

~~Resistor~~ Reactance is tuned out when this term goes to zero.

want it cancels → ICE → ELI

$\omega L - 1/\omega C = 0$

$\omega L = 1/\omega C$

$\omega^2 = 1/LC$

$\boxed{\omega = \frac{1}{\sqrt{LC}}}$

$\boxed{L = \frac{1}{\omega^2 C}}$

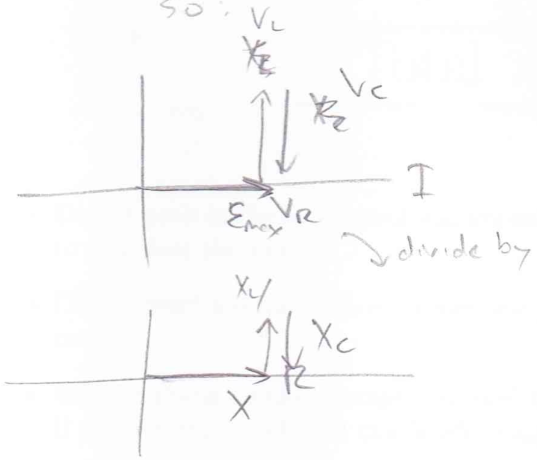
↑ inductance required to tune out reactance.

The current will be in phase!

The reactive components cancel out,

so:

$\frac{V}{R} = IR$



Since $\cancel{V} = Z R$,

current and voltage are in phase.

-8