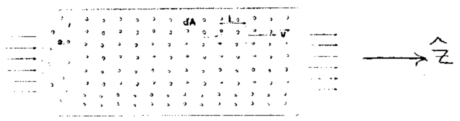


1) A cylindrical particle beam of radius a is made up of electrically-charged particles traveling with a speed v (along the direction of the longitudinal axis). The volume charge density of particles within the beam is given by $\rho(r) = \rho_0 \left(\frac{r}{a}\right)^n$. ($r$ is measured in cylindrical coordinates and n is a positive integer)

- 1a) (5 points) Consider the little sub-volume shown in the diagram. In terms of the quantities you've been given, how much charge is contained within that sub-volume? How long will it take for the charge in the sub-volume to pass through the reference plane? How much will the sub-volume contribute to the current flowing in the beam? From your answers, determine the current density associated with the beam.

$$\left.\begin{aligned} dq &= \rho\, dV \\ dq &= \rho L A \\ T &= L/v \\ dI &= dq/T \\ J &= dI/dA \end{aligned}\right\} \Longrightarrow$$

$$\boxed{\begin{aligned} dq &= \rho_0 L \left(\tfrac{r}{a}\right)^n dA \\ T &= L/v \\ dI &= \rho_0 v \left(r/a\right)^n dA \\ \vec{J} &= \rho_0 v \left(r/a\right)^n \hat{z} \end{aligned}}$$

- 1b) (10 points) How much current passes through a circular loop of radius r if that loop is oriented perpendicular to the beam and is centered on its longitudinal axis? Solve for both $r < a$ and $r > a$. (If you didn't solve the first part, use $J = J_0\left(\frac{r}{a}\right)^n$).

$$I_{enc} = \int \vec{J}(r)\cdot d\vec{A} = \int J(r) 2\pi r\, dr$$

$$(r \le a) \quad I_{enc} = 2\pi \rho_0 v a \int_0^r \left(\tfrac{r}{a}\right)^{n+1} dr$$

$$I_{enc} = \frac{2\pi\rho_0 v a^2}{n+2}\left(\frac{r}{a}\right)^{n+2}$$

$$(r > a) \quad I_{enc} = I_{enc}(a)$$

$$I_{enc} = \frac{2\pi\rho_0 v a^2}{n+2}$$

$$\boxed{I_{enc} = \begin{cases} \dfrac{2\pi a^2 \rho_0 v}{n+2}\left(\dfrac{r}{a}\right)^{n+2} & (r \le a) \\[2ex] \dfrac{2\pi a^2 \rho_0 v}{n+2} & (r > a) \end{cases}}$$

of course, $\rho_0 v \equiv J_0$, $n \equiv \alpha$ if you didn't get the first part

- 1c) (10 points) Find the magnitude and direction of the magnetic field at points inside and outside the beam, assuming the charge carriers are positive.

for a long straight current, ampere's law tells us $B = \frac{\mu_0 I_{enc}(r)}{2\pi r}$, with a direction consistent with the right-hand rule with I_{enc} (in this case, out at the top, in at the bottom, as drawn on the figure)

$$\boxed{\vec{B}(r) = \begin{cases} \dfrac{\mu_0 \rho_0 v a}{n+2}\left(\dfrac{r}{a}\right)^{n+1} \hat{\phi} & (r \le a) \\[2ex] \dfrac{\mu_0 \rho_0 v a^2}{(n+2)\, r}\, \hat{\phi} & (r > a) \end{cases}}$$

where $\hat{\phi}$ is determined by the rhr with the current...

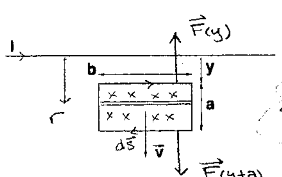
- 1d) (5 points) What force will a particle of charge $+q$ experience if it is located a distance r from the longitudinal axis of the beam? What implications does this have for the beam?

$$\vec{F}_B = q\vec{v}\times\vec{B} = qvB(r)\sin(90^\circ)(-\hat{r})$$

Cylindrical Coordinates

$$\boxed{\vec{F}_B = \frac{q\mu_0\rho_0 v^2 a}{(n+2)}\left(\frac{r}{a}\right)^{n+1}(-\hat{r})}$$

$\vec{F}_B$ is a restoring force (a linear restoring force, if $n=0$!) Particles will oscillate back and forth about the centerline of the beam! (This may explain the beam shape in some glowing tubes of plasma?)



$$B(r) = \frac{\mu_0 I}{2\pi r}$$

2) The diagram above shows a long, straight wire that carries an electrical current I and a rectangular conducting loop of side lengths a and b and resistance R. The loop and the wire are both contained in the plane of the page, and the loop is moving away from the wire (all the time, in the plane of the page) with a velocity $\vec{v}$. You may use y to parameterize the distance between the wire and the closest side of the loop at any instant.

- 2a) (15 points) Find the magnitude and the direction of the induced EMF and the induced current around the conducting loop. Explain, as clearly as you can, how the directions you've obtained are consistent with Lenz's law.

$$\Phi_B = \int \vec{B}\cdot d\vec{A} = \int B(r)\, b\, dr$$

$$\Phi_B = \int_y^{y+a} \frac{\mu_0 I b}{2\pi r}\, dr$$

$$\Phi_B = \frac{\mu_0 I b}{2\pi}\ln\left(\frac{y+a}{y}\right)$$

$$\mathcal{E}_i = -\frac{d\Phi_B}{dt}$$

$$\mathcal{E}_i = -\frac{\mu_0 I b}{2\pi}\left[\frac{1}{y+a} - \frac{1}{y}\right]\frac{dy}{dt}$$

$$\mathcal{E}_i = \frac{\mu_0 I b a}{2\pi y(y+a)} v$$

$$I_i = \mathcal{E}_i/R = \frac{\mu_0 I a b v}{2\pi R y(y+a)}$$

$$\boxed{\begin{aligned} \mathcal{E}_i &= \frac{\mu_0 I a b v}{2\pi y(y+a)} \\ I_i &= \frac{\mu_0 I a b v}{2\pi R y(y+a)} \end{aligned}}$$

Clockwise around the loop, as drawn

The field is getting weaker as the loop moves away - inward flux is decreasing - so we need to create more $\Rightarrow$ rhr requires a CW current

- 2b) (10 points) Find the magnitude and the direction of the external force that is responsible for maintaining the constant velocity of the conducting loop.

$$\vec{F}_{ext} + \vec{F}_{(y)} + \vec{F}_{(a+y)} = m\cancel{\vec{a}}^{\,0}$$

Sum along the radial direction

$$F_{ext,r} - I_i b B(y) + I_i b B(a+y) = 0$$

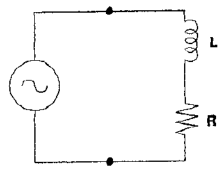
$$F_{ext,r} = \frac{\mu_0^2 I^2 a b^2 v}{4\pi^2 R y (y+a)}\left(\frac{1}{y} - \frac{1}{a+y}\right)$$

$$\boxed{\vec{F}_{ext} = \frac{\mu_0^2 I^2 a^2 b^2 v}{4\pi^2 R\, y^2 (y+a)^2}\,\hat{r}}$$

there's something cool about all those squares :)

- 2c) (5 points) The external agent that is pulling on that loop is obviously doing work. Where is the energy going, if not into changing the kinetic energy of the loop?

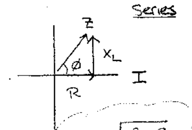
Into moving the charges around the loop, of course :)



3) An inductor (L) and a resistor (R) are connected in series across a source of alternating EMF ($\xi(t) = \xi_{max}\cos(\omega t)$). You may do the following calculations in complex space, but keep in mind, the final answers must all be real.

- 3a) (10 points) What is the impedance of LR combination? What is the amplitude of the current that passes through it? Does the current through this impedance lead or lag the voltage across it? By how much?

Series



$$Z = \sqrt{R^2 + X_L^2}$$
$$X_L = \omega L$$
$$I_{max} = \frac{\mathcal{E}_{max}}{Z}$$
$$\phi = \tan^{-1}\frac{X_L}{R}$$

$$\boxed{Z = \sqrt{(\omega L)^2 + R^2}}$$

$$\boxed{I_{max} = \frac{\mathcal{E}_{max}}{\sqrt{(\omega L)^2 + R^2}}}$$

Current lags Voltage by $\phi = \tan^{-1}\frac{\omega L}{R}$

- 3b) (10 points) What is the voltage amplitude across the resistor? What is the voltage amplitude across the inductor? Will the sum of these voltage amplitudes equal the voltage amplitude of the source? Explain.

$$\Delta V_{R,max} = I_{max} R \Rightarrow \boxed{\Delta V_{R,max} = \frac{\mathcal{E}_{max} R}{\sqrt{(\omega L)^2 + R^2}}}$$
$$\Delta V_{L,max} = I_{max} X_L \qquad \boxed{\Delta V_{L,max} = \frac{\mathcal{E}_{max}\, \omega L}{\sqrt{(\omega L)^2 + R^2}}}$$

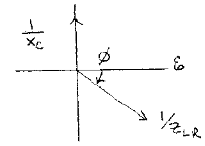
$$\boxed{\Delta V_{R,max} + \Delta V_{L,max} \neq \mathcal{E}_{max}}$$

(They're not in phase - they don't reach their maximums simultaneously

- 3c) (10 points) For a lot of good reasons, it is often desirable to present the source with a purely resistive load. We can tune-out the reactance in the LR network by adding a capacitor in parallel with it (across the two dots in the circuit). What value should the capacitor have? What will be the value of the impedance seen by the source? Will current through this new LRC combination lead or lag the voltage across it? By how much?

Fun! Study the impedance diagram in part a before you start...

Parallel



we want...

$$\frac{1}{X_C} = \frac{1}{Z_{LR}}\sin\phi$$

but $\sin\phi = \frac{X_L}{Z_{LR}}$ (part a)

So...

$$\frac{1}{X_C} = \frac{X_L}{Z_{LR}^2} = \frac{\omega L}{R^2 + (\omega L)^2}$$

$$\omega C = \frac{\omega L}{R^2 + (\omega L)^2}$$

$$\frac{1}{Z} = \frac{1}{Z_{LR}}\cos\phi = \frac{1}{Z_{LR}}\frac{R}{Z_{LR}}$$
$$Z = \frac{Z_{LR}^2}{R}$$

$$\Rightarrow \boxed{C = \frac{L}{R^2 + (\omega L)^2}}$$

$$\boxed{Z = \frac{R^2 + (\omega L)^2}{R}}$$

The load is resistive. Current will be in phase with the Voltage!

← you are, of course, welcome to do this with complex impedances, if you know what you're doing

$$\frac{1}{Z} = -\frac{1}{iX_C} + \frac{1}{R + iX_L}$$
$$\vdots$$

Same result ☺