Name:

Student ID #:

Discussion day, time:

Physics 1C, Spring 2019, Lecture 2
Midterm 1

Time allotted: 50 minutes

No calculators or notes allowed.

No phones out during the exam.

All work must be your own.

Partial credit will be awarded for correct work.

Problem 1	/20
Problem 2	/20
Problem 3	/20
Total	/60

Inspirational quote:

But still, try, for who knows what is possible? —Michael Faraday (1870)

Potentially useful formulas:

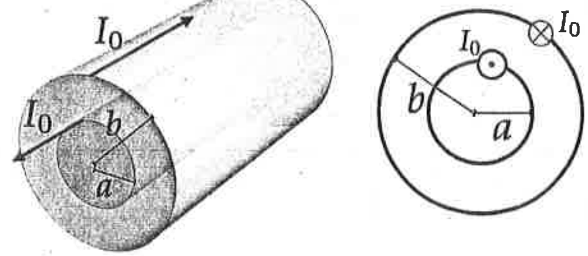
Vacuum magnetic permeability

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}} \tag{1}$$

Field due to a ring (radius a) of current I on axis

$$\mathbf{B}(z) = \frac{\mu_0 I}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}\,\hat{\mathbf{z}} \tag{2}$$

Problem 1. A coaxial cable consists of a long, cylindrical, conducting shell of radius a surrounded by another, coaxial shell of radius $b > a$. The two shells both carry current I_0 distributed evenly over their surface area, but in opposite directions.

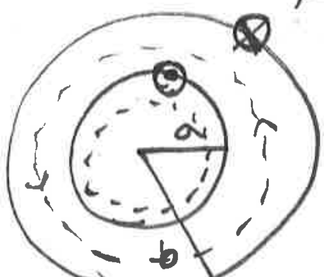


(a) Use Ampere's law to find the magnetic field everywhere as a function of radius.

(b) Calculate the self-inductance L of a length ℓ of such a cable.

(c) A length of this coaxial cable is connected as an inductor to a fixed capacitance C, forming a series LC circuit. Explain how you might vary the geometry of the cable to make the LC oscillation frequency ω_0 large.

(a)



Ampere's law

$r<a$:

$$I_{enc} = 0 \rightarrow \int \vec{B} \cdot d\vec{l} = 0 \rightarrow \boxed{B=0}$$

$a<r<b$:

$$I_{enc} = I_0$$

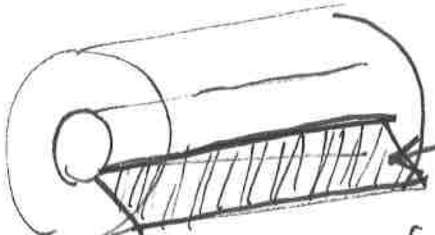
amperian loop is circle of radius r

$$\int \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 I_{enc} = \mu_0 I_0$$

$$\boxed{B = \frac{\mu_0 I_0}{2\pi r}}$$

$r>b$: $\quad I_{enc} = I_0' - I_0 = 0 \rightarrow \boxed{B=0}$

(b) $$L = \frac{\Phi}{I}$$



flux thru here

$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_0^{l} dz \int_a^b dr \left(\frac{\mu_0 I_0}{2\pi r}\right)$$

$$= \frac{\mu_0 I_0 l}{2\pi}\int_a^b dr\, \frac{1}{r}$$

$$= \frac{\mu_0 I_0 l}{2\pi}\log\left(\frac{b}{a}\right)$$

$$L = \Phi / I_0 = \frac{\mu_0 l}{2\pi}\log\left(\frac{b}{a}\right)$$

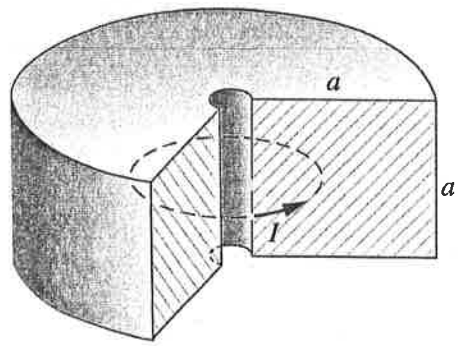
(c) $$\omega_0 = \frac{1}{\sqrt{LC}}$$ to make ω_0 large, make L small

$$L \sim l \log\left(\frac{b}{a}\right)$$

- could make l small, but what if we need a certain length of cable?
- more useful: make b very close to a, then $\log\left(\frac{b}{a}\right)$ becomes small. This makes L small for any length of cable.

Problem 2. Magnetic fields inside good conductors cannot change quickly. We found that current in a simple inductive circuit decays exponentially with characteristic time L/R. In a large conducting body such as the core of the earth, a circuit is not easy to identify. Nevertheless we can find an order-of-magnitude estimate for the decay time τ with some simplifying approximations.

We will model the core of the earth as a conducting "donut" of square cross section (side length a) and total resistance R.



This image shows a portion of the donut cut out to display the cross-section. In your answer, consider the full donut.

(a) As an estimate of the magnitude B of the magnetic field, calculate the field at the center of a conducting ring of radius $a/2$ carrying current I.

(b) Treat the magnetic field in the donut as uniform, with the magnitude you found in (a). Estimate the magnetic field energy U stored in the donut. Treat the donut as a cylinder for the purpose of finding its volume.

(c) Suppose the magnetic energy stored in the donut depends on time roughly as $U \approx U_0 e^{-t/\tau}$, with U_0 constant. Use the fact that the power is

$$\frac{dU}{dt} = -I^2 R \tag{3}$$

and your answer to (b) to obtain an expression for τ in terms of a, R, and fundamental constants. Verify that your expression is dimensionally correct.

(a) From formula sheet:

$$B_{ring} = \frac{\mu_0 I}{2} \frac{r^2}{(z^2+r^2)^{3/2}}$$

at center, $z=0$. For us, $r = a/2$

$$B_{ring} = \frac{\mu_0 I}{2} \frac{(a/2)^2}{((a/2)^2)^{3/2}} = \frac{\mu_0 I}{2}\left(\frac{1}{a/2}\right)$$

$$\boxed{= \frac{\mu_0 I}{a}}$$

(b)
$$u = \frac{1}{2\mu_0} B^2$$
$$= \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{a}\right)^2 = \frac{\mu_0}{2}\frac{I^2}{a^2}$$

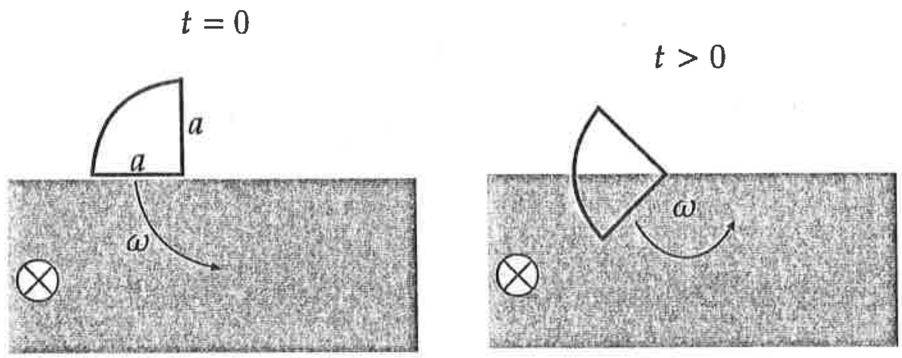
u is uniform, so $U = uV$

$V =$ volume of cylinder
$$= (\pi a^2)\, a$$
$$= \pi a^3$$
$$U = \frac{\mu_0}{2}\frac{I^2}{a^2}\pi a^3 = \boxed{\frac{\mu_0 I^2 \pi a}{2}}$$

(c) $U = U_0 e^{-t/\tau}$ so
$$\frac{dU}{dt} = -\frac{1}{\tau} U = -I^2 R$$
$$\Rightarrow \tau = \frac{U}{I^2 R} = \frac{\mu_0 I^2 \pi a}{2 I^2 R}$$
$$= \boxed{\frac{\mu_0 \pi a}{2R}}$$

dimensions:
$$[\text{time}] = \frac{\left[\frac{\text{inductance}}{\text{length}}\right][\text{length}]}{[\text{resistance}]}$$
$$= \left[\frac{\text{inductance}}{\text{resistance}}\right] = [\text{time}] \checkmark$$

Problem 3. A conducting wire loop in the shape of a quadrant of a circle with radius a rotates at constant angular velocity $d\theta/dt = \omega$ as shown in the figure below. The shaded area indicates a constant, uniform magnetic field of magnitude B directed into the page.



$\mathbf{B} = -B\,\hat{\mathbf{z}}$

(a) Calculate the emf $\mathcal{E}(t)$ around the wire as a function of time for one full period, $0 \le t < 2\pi/\omega$. Your answer will be a piecewise function with four time segments. The area of a circular sector of radius r subtending angle θ is Area $= r^2\theta/2$.

(b) For some time segments of its motion, the loop will experience a magnetic force. Without detailed calculations, give the direction of this force, if any, during each segment of time.

(a) ① $0 \le t < \frac{\pi}{2\omega}$: loop entering field

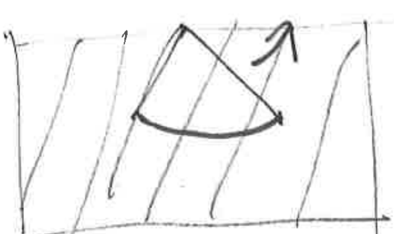
~~flux through loop increasing~~

$$\Phi = \vec{B}\cdot\vec{A} = -BA \rightarrow \frac{d\Phi}{dt} = -B\frac{dA}{dt}$$

$$A = \frac{a^2\theta}{2}, \quad \frac{dA}{dt} = \frac{a^2}{2}\frac{d\theta}{dt} = \frac{a^2\omega}{2}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = \boxed{\frac{Ba^2\omega}{2}}$$

② $\frac{\pi}{2\omega} \le t < \frac{\pi}{\omega}$: loop completely in field



$$\frac{d\Phi}{dt} = 0 \Rightarrow \boxed{\mathcal{E} = 0}$$

③ $\frac{\pi}{\omega} \le t < \frac{3\pi}{2\omega}$: loop exiting field

Now $\frac{dA}{dt}$ is negative, b/c area in field region decreasing.

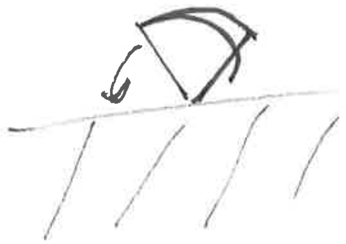
$$\frac{d\Phi}{dt} = -B\frac{dA}{dt}, \quad \frac{dA}{dt} = -\frac{a^2\omega}{2}$$

(same as 1st time segment, but negative)

$$\mathcal{E} = -\frac{d\Phi}{dt} = \boxed{-\frac{Ba^2\omega}{2}}$$

④ $\frac{3\pi}{2\omega} \le t < \frac{2\pi}{\omega}$: loop completely out of field

$$\frac{d\Phi}{dt} = 0 \Rightarrow \boxed{\mathcal{E} = 0}$$



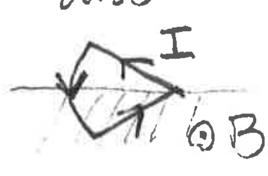
(b) Induced emf opposes change in flux.

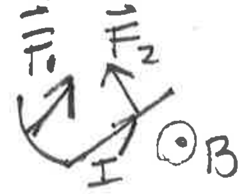
For ②, ④, no force.

For ①, force points $\boxed{\text{away from field region}}$, opposing $\Delta\Phi$ caused by entering field

For ③, force points $\boxed{\text{into field region}}$, opposing $\Delta\Phi$ caused by leaving field.

(could also argue from $\vec{F} = I\vec{l}\times\vec{B}$





$\vec{F}_{net} = \uparrow$, etc.)