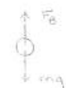
The buoyant force exerted on a hot-air balloon (of average mass-density ρ_{gas}) by the air it is immersed in is given by $F_B = \rho_0 V g e^{-y/\lambda}$, where ρ_0 is the mass-density of air at ground-level, V is the volume of the balloon, y is the height of the balloon above ground-level and λ is some scaling constant. The values of ρ_{gas}, ρ_0, V, g, and λ are all known.

- a) (10 pts) Using mechanics, **obtain a differential equation** (in terms of y) that describes the vertical motion of the hot-air balloon. **Find the height** at which the balloon will be in equilibrium (y_{eq}).
- b) (10 pts) Intuitively, it seems reasonable to assume that the balloon is in stable equilibrium at the height found in part a - that is, if we displace it by a small amount, we should observe a net restorative force that tries to bring it back to the equilibrium position. Hmmm... Substitute $y = y_{eq} + \delta y$ into the differential equation from part a to **obtain a new differential equation in terms of displacement from equilibrium** (δy) and show that for small displacements from equilibrium, the balloon executes simple harmonic motion about equilibrium [hint: $e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! \ldots$]. **What is the angular frequency of the resulting oscillation?**
- c) (5 pts) In October, hundreds of hot-air balloons will lift-off from Albuquerque as part of their International Balloon Festival. How will the equilibrium heights reached by the balloons vary from balloon to balloon? How will the angular frequency of oscillation vary from balloon to balloon?

$m = \rho_{gas} V$

a) $\Sigma F_y = m a_y$



$\rho_0 V g e^{-y/\lambda} - \rho_{gas} V g = \rho_{gas} V \frac{d^2 y}{dt^2}$

$$\boxed{\frac{d^2 y}{dt^2} - \frac{\rho_0}{\rho_{gas}} g e^{-y/\lambda} = -g}$$

in equilibrium $\frac{d^2 y}{dt^2} = 0$

$-\frac{\rho_0}{\rho_{gas}} g e^{-y/\lambda} = -g$

$e^{-y/\lambda} = \frac{\rho_{gas}}{\rho_0}$

$$\boxed{y_{eq} = \lambda \ln\left(\frac{\rho_0}{\rho_{gas}}\right)}$$

EXTRA PAGE

b) $y = y_{eq} + \delta y$

$\frac{d^2 y}{dt^2} = \frac{d^2 \delta y}{dt^2}$ $\Rightarrow$ $\frac{d^2 \delta y}{dt^2} - \frac{\rho_0}{\rho_{gas}} g e^{-\frac{1}{\lambda}(\lambda \ln(\rho_0/\rho_{gas}) + \delta y)} = -g$

$$\boxed{\frac{d^2 \delta y}{dt^2} - g e^{-\delta y/\lambda} = -g}$$

$(e^x \approx 1 + x + \ldots)$ for small δy

$\frac{d^2 \delta y}{dt^2} - g\left(1 - \frac{\delta y}{\lambda}\right) = -g$

"sho" $\longrightarrow$ $\boxed{\frac{d^2 \delta y}{dt^2} + \frac{g}{\lambda} \delta y = 0}$

$$\boxed{\omega = \sqrt{g/\lambda}}$$

c) The equilibrium heights depend on the average mass density of the balloons — denser balloons will find lower equilibrium heights.

ω does not depend on the balloon parameters [illegible]