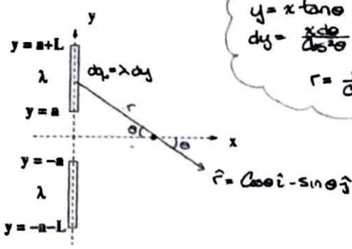
by symmetry:
$E_y = 0$
$E_z = 0$

$dE_x = dE\cos\theta$

$dE_x = \frac{dq\, x}{4\pi\epsilon_0 r^3}$

$dE_x = \frac{\lambda\, dy\, x}{4\pi\epsilon_0 r^3}$



$y = x\tan\theta$
$dy = \frac{x\, d\theta}{\cos^2\theta}$
$r = \frac{x}{\cos\theta}$

1) Two identical, non-conducting rods of length L and linear charge density λ are placed along the y-axis so that the origin lies at the center of the gap of length $2a$ that sits between them, as shown.

- 1a) (15 points) Find the (vector) electric field for points along the positive x-axis.

$$dE_x = \frac{\lambda\, dy\, x}{4\pi\epsilon_0 r^3} = \frac{\lambda x^2 d\theta \cos^3\theta}{4\pi\epsilon_0 \cos^2\theta\, x^3} = \frac{\lambda}{4\pi\epsilon_0 x}\, d\theta \cos\theta$$

$$E_x = \int dE_x = \frac{\lambda}{4\pi\epsilon_0 x}\left[\int_{\theta_{-a-L}}^{\theta_{-a}} d\theta\cos\theta + \int_{\theta_a}^{\theta_{a+L}} d\theta\cos\theta\right]$$

$$E_x = \frac{2\lambda}{4\pi\epsilon_0 x}\sin\theta\Big|_{\theta_a}^{\theta_{a+L}}$$

$$E_x = \frac{\lambda}{2\pi\epsilon_0 x}\left[\frac{a+L}{\sqrt{(a+L)^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right]$$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\left[\frac{a+L}{\sqrt{(a+L)^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right]\hat{\imath}}$$

- 1b) (5 points) Under what conditions would you expect a point charge q, released at rest on the x-axis, to undergo simple harmonic motion about the origin?

$$\boxed{\begin{array}{l}\text{if } q\lambda = -|q\lambda| \text{ and} \\ \text{if } x \ll a \\ q \text{ should undergo SHO}\end{array}}$$

- 1c) (10 points) Suppose a point charge q (of mass m) is released under the conditions given above. Show that it will indeed execute simple harmonic motion and find the angular frequency of that motion

$$F_x = qE_x = \frac{\lambda q}{2\pi\epsilon_0 x}\left[\left(1+\frac{x^2}{(a+L)^2}\right)^{-1/2} - \left(1+\frac{x^2}{a^2}\right)^{-1/2}\right]$$

$x \ll a < a+L$, so...

$$F_x \simeq \frac{q\lambda}{2\pi\epsilon_0 x}\left[1 - \frac{1}{2}\frac{x^2}{(a+L)^2} - 1 + \frac{1}{2}\frac{x^2}{a^2}\right]$$

$$F_x \simeq \frac{q\lambda}{4\pi\epsilon_0}\left[\frac{1}{a^2} - \frac{1}{(a+L)^2}\right]x\,\hat{\imath}$$

if $q\lambda = -|q\lambda|$...

$$\boxed{\begin{array}{l} F_x \simeq \frac{-|q\lambda|}{4\pi\epsilon_0}\frac{L(L+2a)}{a^2(a+L)^2}\, x \\ \omega_0 = \sqrt{\frac{-q\lambda L(L+2a)}{4\pi\epsilon_0 m a^2(a+L)^2}}\end{array}}$$

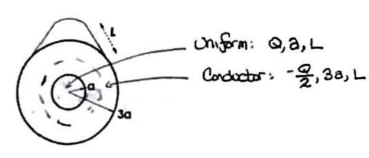
LINEAR RESTORING FORCE

if $F_x = -kx$
$\omega_0 = \sqrt{k/m}$

Cylindrical symmetry:

$$\vec{E}(r) = \frac{q_{in}(r,x)/x}{2\pi\epsilon_0 r}\hat{r}$$

(Gauss)



2) An electric charge Q is uniformly distributed within a long cylinder of radius a and length L ($L >>> a$). This cylinder is, in turn, surrounded by a coaxial cylindrical conducting shell that fills the region from $r = a$ to $r = 3a$ (where r is the radial distance measured from the symmetry axis of the first distribution). The conductor carries an electric charge equal to $-Q/2$.

- 2a) (15 points) Find the (vector) electric field for all values of r. For this part and all that follow, you may assume the points you are looking at are far from either end of the long cylindrical distribution.

$(r<a)$ Uniform $\Rightarrow \rho = \rho \Rightarrow \frac{Q}{\pi a^2 L} = \frac{q_{in}(r,x)}{\pi r^2 x} \Rightarrow \frac{q_{in}(r,x)}{x} = \frac{Q}{L}\frac{r^2}{a^2}$

$(a<r<3a)$ inside physical body of Conductor : $\vec{E} = 0$

$(3a<r)$ $q_{in}(r,x) = \left[Q + \left(-\frac{Q}{2}\right)\right]\frac{x}{L} \Rightarrow \frac{q_{in}(r,x)}{x} = \frac{Q}{2L}$

x is a fraction of the total length, $q_{in}(r,x)$ is a fraction of total charge

$\Rightarrow$ Using Gauss' Law:

$$\boxed{\begin{array}{ll} \vec{E} = \frac{Qr}{2\pi\epsilon_0 L a^2}\hat{r} & (r<a) \\ \vec{E} = 0 & (a<r<3a) \\ \vec{E} = \frac{Q}{4\pi\epsilon_0 L r}\hat{r} & (3a<r)\end{array}}$$

$\Delta V(2a,r) = V(r) - V(2a) = V(r) - V_0 = -\int_{2a}^{r} E\,dr$ (cyl. sym)

- 2b) (10 points) Find the electric potential at all values of r if the potential at $r = 2a$ is equal to V_0.

$$V(r) = V_0 - \int_{2a}^{r} E\,dr$$

$(r<a)$ $V(r) = V_0 - \int_{2a}^{a} 0\,dr - \int_{a}^{r}\frac{Qr\,dr}{2\pi\epsilon_0 L a^2} = V_0 + \frac{Q}{4\pi\epsilon_0 L}\left(1 - \frac{r^2}{a^2}\right)$

$(a<r<3a)$ $V(r) = V_0 - \int_{2a}^{a} 0\,dr = V_0$

$(3a<r)$ $V(r) = V_0 - \int_{2a}^{3a} 0\,dr - \int_{3a}^{r}\frac{Q\,dr}{4\pi\epsilon_0 L r} = V_0 - \frac{Q}{4\pi\epsilon_0 L}\ln\left(\frac{r}{3a}\right)$

$$\boxed{\begin{array}{ll} V(r) = V_0 + \frac{Q}{4\pi\epsilon_0 L}\left(1 - r^2/a^2\right) & (r<a) \\ V(r) = V_0 & (a<r<3a) \\ V(r) = V_0 - \frac{Q}{4\pi\epsilon_0 L}\ln\left(\frac{r}{3a}\right) & (3a<r)\end{array}}$$

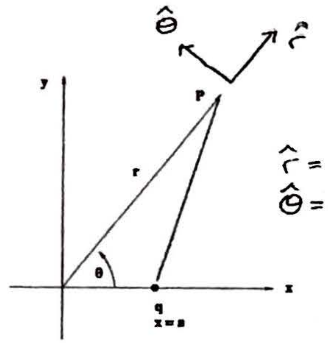
- 2c) (5 points) How much work would one have to do to move a test-charge of electric charge q from $r = 6a$ to $r = \frac{1}{2}a$?

$V_i = V(6a) = V_0 - \frac{Q}{4\pi\epsilon_0 L}\ln(2)$

$V_f = V\left(\frac{a}{2}\right) = V_0 + \frac{Q}{4\pi\epsilon_0 L}\cdot\frac{3}{4}$

$W_{ext} = \Delta U_e = q\,\Delta V$

$$\boxed{W_{ext} = \frac{Qq}{4\pi\epsilon_0 L}\left[\frac{3}{4} + \ln(2)\right]}$$



$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

3) A point charge q is placed on the x-axis, a distance a from the origin. Point P is located in the x,y-plane a distance r from the origin, along a line rotated clockwise from the x-axis by an angle θ.

- **3a) (5 points)** Find the (exact) electric potential at point P.

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}}$$

- **3b) (10 points)** Use Taylor's theorem (to first order) to approximate the potential in the limit $r \gg a$. You should have two terms.

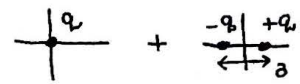
$$V \simeq \frac{q}{4\pi\epsilon_0 r}\left(1 - 2\frac{a}{r}\cos\theta\right)^{-1/2}$$

$$\boxed{V \simeq \frac{q}{4\pi\epsilon_0 r}\left(1 + \frac{a}{r}\cos\theta\right)}$$

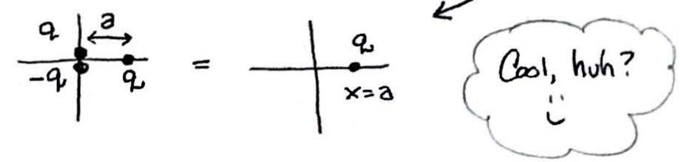
- **3c) (5 points)** Interpret the two terms. It may be useful to sketch the corresponding charge distribution for each. Overlap those sketches into a third sketch and see if you can make sense of what you're seeing.

$V_1 = \frac{q}{4\pi\epsilon_0 r}$ ← monopole term : moment $= q$

$V_2 = \frac{qa\cos\theta}{4\pi\epsilon_0 r^2}$ ← dipole term : moment $|\vec{p}| = qa$ oriented θ from $\vec{r}$ (along + x-axis)



→ a << r, so we can slide the dipole over a bit and not change anything



- **3d) (10 points)** Still out at $r \gg a$, find the x-, y- and z-components of the electric field vector.

$$E_r = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2} + \frac{2a}{r^3}\cos\theta\right]$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{q}{4\pi\epsilon_0}\frac{a}{r^3}\sin\theta$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\left(\frac{1}{r^2} + \frac{2a}{r^3}\cos\theta\right)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + \frac{a}{r^3}\sin\theta(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\right]$$

$$E_x = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2}\cos\theta + \frac{a}{r^3}(2\cos^2\theta - \sin^2\theta)\right)$$

$$E_y = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2}\sin\theta + \frac{a}{r^2}3\sin\theta\cos\theta\right)$$

$$E_z = 0$$

Just a superposition of monopole & dipole fields!