MT2 Physics 1B W16

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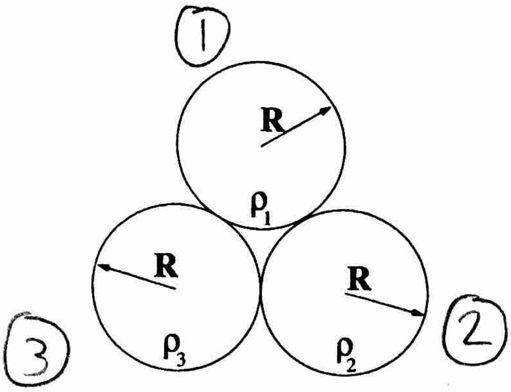
Full Name (Signature) [signature]

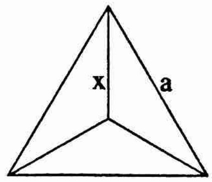
Student ID Number 704574468

Seat Number ______

Problem	Grade
1	30 /30
2	18 /30
3	23 /30
Total	71 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**





$a^2 = 3x^2$

1) Three spheres of identical radius R (but different, uniform charge densities ρ_1, ρ_2 and ρ_3) are arranged so that they touch one another as shown in the diagram above. [The diagram at the right is a hint for later].

+10

- 1a) (10 points) How much work will it take to assemble these spheres into the arrangement shown? [Assume the spheres themselves have already been assembled - that is, neglect the self-energy of each sphere].

find total potential energy:

$$Q_1 = \frac{4}{3}\pi \rho_1 R^3 \qquad U_{12} = \frac{kQ_1Q_2}{2R} = \frac{1}{2R}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{16}{9}\pi^2\rho_1\rho_2R^6 = \frac{2\pi\rho_1\rho_2R^5}{9\varepsilon_0}$$

$$Q_2 = \frac{4}{3}\pi \rho_2 R^3 \qquad U_{23} = \frac{2\pi\rho_2\rho_3R^5}{9\varepsilon_0}$$

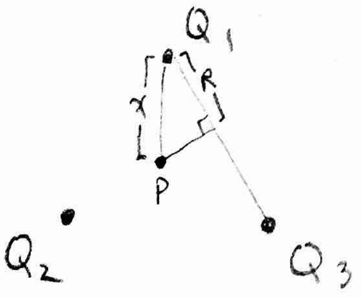
$$Q_3 = \frac{4}{3}\pi \rho_3 R^3 \qquad U_{31} = \frac{2\pi\rho_3\rho_1R^5}{9\varepsilon_0}$$

$$U_{tot} = \frac{2\pi R^5}{9\varepsilon_0}\left(\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1\right)$$

+10

- 1b) (10 points) What is the electric potential at the center of the arrangement?

treat spheres as three point charges:



$$x = \frac{R}{\sin 60^\circ} = \frac{2R}{\sqrt{3}}$$

$$V_1 = \frac{kQ_1}{x} = \frac{1}{\cancel{4\pi}\varepsilon_0}\cdot\frac{\cancel{\sqrt{3}}}{2\cancel{R}}\cdot\frac{\cancel{4}\cancel{\pi}}{\cancel{3}\sqrt{3}}\rho_1 R^{\cancel{3}\,2} = \frac{\rho_1 R^2}{2\sqrt{3}\,\varepsilon_0}$$

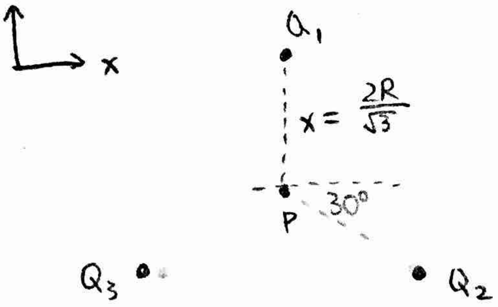
$$V_2 = \frac{\rho_2 R^2}{2\sqrt{3}\,\varepsilon_0}$$

$$V_3 = \frac{\rho_3 R^2}{2\sqrt{3}\,\varepsilon_0}$$

$$V_{tot} = \frac{R^2}{2\sqrt{3}\,\varepsilon_0}\left(\rho_1 + \rho_2 + \rho_3\right)$$

+10

- 1c) (10 points) What is the electric field at the the center of the arrangement?



$\vec{E}_1$:

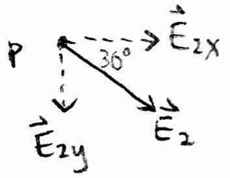
$$\vec{E}_{1x} = 0$$

$$\vec{E}_{1y} = \frac{kQ_1}{x^2}\hat{j} = \frac{1}{\cancel{4\pi}\varepsilon_0} \cdot \frac{\cancel{3}}{4R^{\cancel{2}}} \cdot \cancel{\tfrac{4}{3}}\cancel{\pi}\rho_1 R^{\cancel{3}}\hat{j}$$

$$= \vec{E}_1 \qquad = \frac{\rho_1 R}{4\varepsilon_0}\hat{j}$$

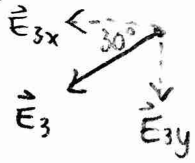
$\vec{E}_2$:

$$\vec{E}_{2x} = E_2 \cos 30^\circ \hat{i} = \frac{\rho_2 R}{4\varepsilon_0} \cdot \frac{\sqrt{3}}{2}\hat{i}$$



$$\vec{E}_{2y} = -E_2 \sin 30^\circ \hat{j} = \frac{-\rho_2 R}{4\varepsilon_0} \cdot \frac{1}{2}\hat{j}$$

$\vec{E}_3$:

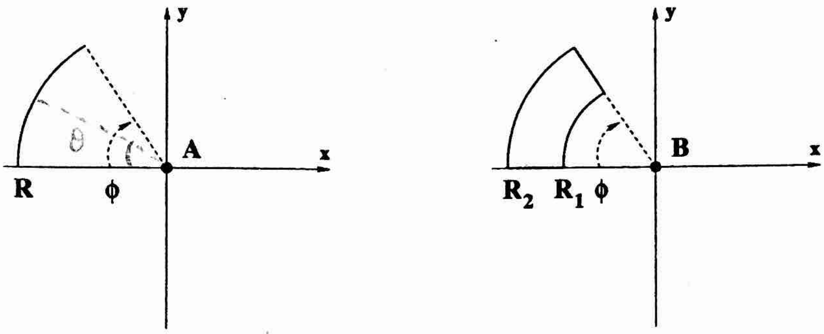


$$\vec{E}_{3x} = -E_3 \cos 30^\circ \hat{i} = -\frac{\rho_3 R}{4\varepsilon_0} \cdot \frac{\sqrt{3}}{2}\hat{i}$$

$$\vec{E}_{3y} = -E_3 \sin 30^\circ \hat{j} = -\frac{\rho_3 R}{4\varepsilon_0} \cdot \frac{1}{2}\hat{j}$$

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = \frac{\rho_1 R}{4\varepsilon_0}\hat{j} - \frac{\rho_2 R}{4\varepsilon_0} \cdot \frac{1}{2}\hat{j} - \frac{\rho_3 R}{4\varepsilon_0} \cdot \frac{1}{2}\hat{j} + \frac{\rho_2 R}{4\varepsilon_0} \cdot \frac{\sqrt{3}}{2}\hat{i} - \frac{\rho_3 R}{4\varepsilon_0} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{R}{4\varepsilon_0}\left(\rho_1 - \frac{1}{2}\rho_2 - \frac{1}{2}\rho_3\right)\hat{j} + \frac{R}{4\varepsilon_0}\left(\frac{\sqrt{3}}{2}\rho_2 - \frac{\sqrt{3}}{2}\rho_3\right)\hat{i}$$



- 2a) (10 points) A thin nonconducting rod that carries an electric charge q (uniformly distributed) is bent to form a circular arc of radius R that subtends an angle ϕ as shown in the diagram on the left. Find the electric field (vector) at point A (located at the center of curvature of the arc).

$$\vec{E}_x = \int_0^\phi dE \cos\theta \,\hat{\imath}$$

$$= \int_0^\phi \underbrace{\frac{d\theta}{\phi} \cdot q}_{dq} \cdot \frac{1}{R^2} \cos\theta \,\hat{\imath}$$

$$= \frac{q}{\phi R^2} \int_0^\phi \cos\theta \, d\theta \,\hat{\imath}$$

$$= \frac{q}{\phi R^2} \sin\theta \,\hat{\imath} \Big|_0^\phi$$

$$= \frac{q \sin\phi \,\hat{\imath}}{\phi R^2}$$

$$\vec{E}_y = -\int_0^\phi dE \sin\theta \,\hat{\jmath}$$

$$= -\frac{q}{\phi R^2} \int_0^\phi \sin\theta \, d\theta \,\hat{\jmath}$$

$$= \frac{q}{\phi R^2} \cos\theta \,\hat{\jmath} \Big|_0^\phi$$

$$= \frac{q(\cos\phi - 1)}{\phi R^2} \hat{\jmath}$$

+9

$$\vec{E}_{tot} = k \frac{q}{\phi R^2} \left(\sin\phi \,\hat{\imath} + (\cos\phi - 1) \hat{\jmath} \right)$$

- 2b) (10 points) Now consider the diagram shown on the right. Charge is spread over a wedge defined by the angle ϕ between the radial distances R_1 and R_2 (as shown) with an area charge density

$$\sigma(r) = \frac{4Q}{\phi(R_2^4 - R_1^4)} r^2$$

where r is the radial distance from the B (located at the center of curvature of the defining arcs). Find the electric field (vector) at B.

$$\vec{E} = \int_{R_1}^{R_2} d\vec{E} = \int_{R_1}^{R_2} \frac{dq}{\phi r^2} \left(\sin\phi \,\hat{\imath} + (\cos\phi - 1) \hat{\jmath} \right)$$

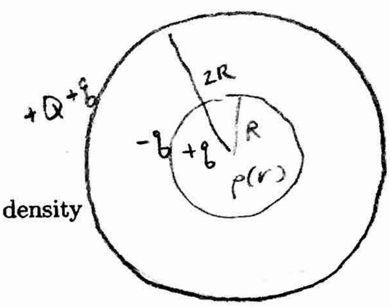
$$= \int_{R_1}^{R_2} \frac{\phi R \cdot dr \cdot \sigma(r)}{\phi r^2} \left(\sin\phi \,\hat{\imath} + (\cos\phi - 1) \hat{\jmath} \right)$$

+8

• 2b) *(continued...)*

• 2c) (10 points) Find the electric potential produced by the wedge at point B relative to a point infinitely-distant from the wedge.

$$V = \int_{R_1}^{\infty} \vec{E}\, dr$$



3) A spherical charge distribution of radius R carries a volume charge density

$$\rho(r) = \rho_0 \, (1 - \frac{r^3}{R^3})$$

It is surrounded by a concentric spherical conducting shell that extends from $r = R$ to $r = 2R$ and carries an excess charge Q.

- 3a) (10 points) Find the charge inside a concentric sphere of radius r, for all values of r. Also, find the surface charge densities on the inner and outer surfaces of the conducting shell.

$r < R$:

$$q(r) = \int_0^r \rho_0 (1 - \frac{s^3}{R^3}) \cdot 4\pi s^2 ds$$

$$= 4\pi\rho_0 \int_0^r (s^2 - \frac{s^5}{R^3}) ds$$

$$= 4\pi\rho_0 \left(\frac{1}{3} r^3 - \frac{1}{6} \cdot \frac{r^6}{R^3}\right)$$

2

$$\sigma_{inner} = - \frac{\frac{2}{3}\pi\rho_0 R^3}{4\pi R^2} = - \frac{\rho_0 R}{6}$$

2

$$\sigma_{outer} = + \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{4\pi (2R)^2}$$

$R \leq r < 2R$:

$$q(r) = 0$$

2

2

$r \geq 2R$:

$$q(r) = \int_0^R \rho(s) \cdot 4\pi s^2 ds \quad + Q$$

$$= 4\pi\rho_0 \left(\frac{1}{3} r^3 - \frac{1}{6} \cdot \frac{r^6}{R^3}\right)\Big|_0^R + Q$$

$$= \frac{2}{3}\pi\rho_0 R^3 + Q$$

2

- 3b) (10 points) Find the electric field as a function of the radial distance from the center of the charge distribution (r) for all values of r.

$r < R$: gaussian surface at radius r

$$\int \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{4\pi\rho_0}{\varepsilon_0} \left(\frac{1}{3} r^3 - \frac{1}{6} \cdot \frac{r^6}{R^3}\right)$$

$$\vec{E} = \frac{\rho_0}{\varepsilon_0} \left(\frac{1}{3} r - \frac{1}{6} \cdot \frac{r^4}{R^3}\right) \hat{r}$$

$r \geq 2R$:

$$\int \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{\varepsilon_0}$$

$$\vec{E} = \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{4\pi\varepsilon_0 r^2} \hat{r}$$

$R \leq r < 2R$:

no charge within gaussian surface

$$\vec{E} = 0$$

10

- 3c) (10 points) If the electric potential within the conductor is given as V_0, find the potential as a function of the radial distance from the center of the charge distribution (r) for all values of r.

$r < R$:

$$V(r) = V_{rR} + V_{R2R} + V_{2R\infty}$$

$$= \int_r^R \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3}s - \frac{1}{6}\cdot\frac{s^4}{R^3}\right)ds + V_0 + \int_{2R}^{\infty} \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{6}s^2 - \frac{1}{30}\cdot\frac{s^5}{R^3}\right)\Big|_r^R + V_0 - \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{4\pi\varepsilon_0 r}\Big|_{2R}^{\infty}$$

$$= \frac{\rho_0}{\varepsilon_0}\cdot\frac{2}{15}R^2 - \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{6}r^2 - \frac{1}{30}\cdot\frac{r^5}{R^3}\right) + V_0 + \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{8\pi\varepsilon_0 R}$$

2

$R \leq r < 2R$

$$V(r) = V_{r2R} + V_{2R\infty}$$

$$= V_0 + \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{8\pi\varepsilon_0 R}$$

0

$r \geq 2R$

$$V(r) = V_{r\infty}$$

$$= \frac{\frac{2}{3}\pi\rho_0 R^3 + Q}{4\pi\varepsilon_0 r}$$