MT2 Physics 1B S15-5

Full Name (Printed) ____________________

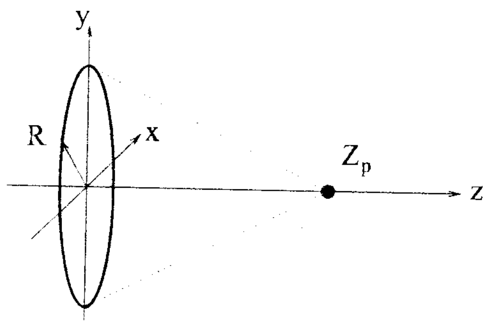
Full Name (Signature) ____________________

Student ID Number ____________________

Seat Number ____________________

Problem	Grade
1	20 /30
2	26 /30
3	13 /30
Total	59 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**



- 1a) (5 points) If the circle shown represents a uniform **ring** of charge Q and radius R, what is the resultant electric field at z_p?

$\vec{E}_{total} = \int d\vec{E}$ ~~Sin~~ By symmetry, the y-component of the electric field due to each infinitesimal point charge on the ring cancels out. Hence $\int d\vec{E} = \int dE_x = \int |d\vec{E}| \cos\theta = \int \frac{dq \cos\theta}{4\pi\epsilon_0 (R^2+z_p^2)}$

$dq = \lambda ds = \lambda R\, d\theta$, $\cos\theta = \frac{z_p}{\sqrt{z_p^2+R^2}}$

$\Rightarrow \int \frac{k\, dq \cos\theta}{R^2+z_p^2}$ where $k = \frac{1}{4\pi\epsilon_0}$ $= k\int \frac{\lambda R z_p}{(R^2+z_p^2)^{\frac{3}{2}}} d\theta = k\lambda R \int_0^{2\pi} \frac{z_p}{(R^2+z_p^2)^{\frac{3}{2}}} d\theta$

$= \frac{2\pi k \lambda R z_p}{(R^2+z_p^2)^{\frac{3}{2}}} = \frac{kQz_p}{(R^2+z_p^2)^{\frac{3}{2}}}$ (since $Q = 2\pi R\lambda$)

$\therefore \vec{E}_{z_p} = \frac{kQz_p}{(R^2+z_p^2)^{\frac{3}{2}}} \hat{k}$

5

- 1b) (5 points) If the circle shown represents a uniform **disk** of charge Q and radius R, what is the resultant electric field at z_p?

$\vec{E}_{disk} = \int d\vec{E}_{ring} = \int \frac{k\, dq\, z_p}{(R^2+z_p^2)^{\frac{3}{2}}}$ where $dq = \sigma 2\pi r\, dr$

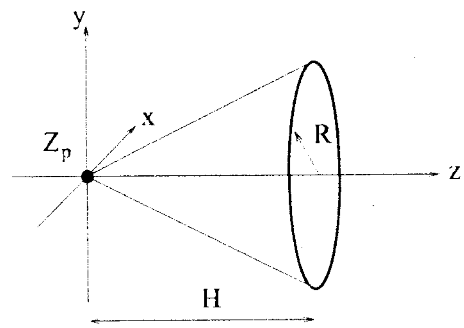
$= 2\pi\sigma k \int_0^R \frac{r z_p\, dr}{(R^2+z_p^2)^{\frac{3}{2}}} = 2\pi\sigma k z_p \int_0^R \frac{r\, dr}{(r^2+z_p^2)^{\frac{3}{2}}} = \pi\sigma k z_p \int_0^R \frac{2r\, dr}{(r^2+z_p^2)^{\frac{3}{2}}}$

$= \pi\sigma k z_p \left[\frac{(r^2+z_p^2)^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_0^R \right] = -2\pi\sigma k z_p \left[(r^2+z_p^2)^{-\frac{1}{2}} \Big|_0^R \right]$

$= -2\pi\sigma k z_p \left[\frac{1}{\sqrt{R^2+z_p^2}} - \frac{1}{z_p} \right] \hat{k}$

$= 2\pi\sigma k z_p \left[\frac{1}{z_p} - \frac{1}{\sqrt{R^2+z_p^2}} \right] \hat{k}$

5



- 1c) (20 points) A charge distribution occupies the volume of a right circular cone of base-radius R and height H that is oriented so that its apex is on the origin and its longitudinal symmetry axis lies along the $+z$-axis with the base intersecting $z = H$, as shown. Assuming the charge distribution has a volume density given by

$$\rho(z) = \frac{(n+3)\,Q}{\pi R^2\, H^{n+1}}\; z^n$$

find the (vector) electric field at the origin.

$$\vec{E}_{total} = \int d\vec{E}_{disk} = \int 2\pi \frac{dq}{\pi r^2} k \left[\frac{1}{z} - \frac{1}{\sqrt{r^2+z^2}}\right] \qquad dq = \rho \pi r^2\, dz$$

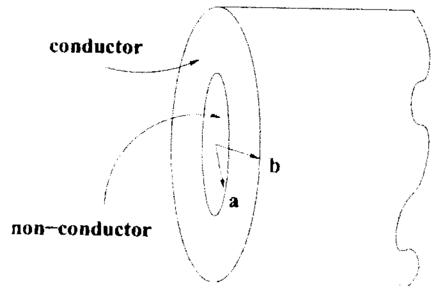
$$\sigma = \frac{dq}{\pi r^2}$$

$$= \int 2\pi \rho\, dz\, k \left[\frac{1}{z} - \frac{1}{\sqrt{r^2+z^2}}\right]$$

$$= 2\pi k \int z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2+H^2}}\right]\left[\frac{(n+3)Q}{\pi R^2 H^{n+1}} z^n\right] dz$$

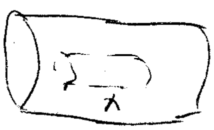
$$= 2\pi k \int_0^H \left(1 - \frac{z}{\sqrt{z^2+H^2}}\right)\left(\frac{(n+3)Q}{\pi R^2 H^{n+1}} z^n\right) dz$$

10



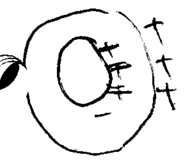
2) An infinitely-long, non-conducting cylinder of radius a carries a uniform volume charge-density ρ, The non-conducting cylinder is, in turn, surrounded by a concentric, neutral conducting cylinder of inner-radius a and outer-radius b.

- 2a) (10 points) Find the amount of charge contained in a concentric cylinder of radius r, length L, for $r < a$, $a < r < b$ and $r > b$.



$(r<a)$ $\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E(2\pi r x) = \frac{\rho(\pi r^2 x)}{\varepsilon_0}$ $\quad \oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$]

$$E = \frac{\rho \pi r^2 x}{2\pi r x \varepsilon_0} = \frac{\rho r}{2\varepsilon_0}$$



$(r<a)$ $q_{in} = \rho(\pi r^2 L) = \rho \pi r^2 L$ ✓

$(a<r<b)$ $q_{in} = 0$ ✓

$(r>b)$ $q_{in} =$ $\rho(\pi a^2 L) = \rho \pi a^2 L$ ✓

- 2b) (10 points) Find the electric field (vector) at all points inside and outside the distribution, as a function of distance from the longitudinal symmetry axis.

$(r<a)$ $\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E(2\pi r x) = \frac{\rho(\pi r^2 x)}{\varepsilon_0}$

$\vec{E} = \frac{\rho \pi r^2 x}{2\pi r x \varepsilon_0} = \frac{\rho r}{2\varepsilon_0}\hat{r}$ ✓

$(a<r<b)$ $\vec{E} = 0$ ✓

$(r>b)$ $E(2\pi r x) = \frac{\rho(\pi a^2 x)}{\varepsilon_0}$

$\vec{E} = \frac{\rho(\pi a^2 x)}{2\pi r x \varepsilon_0} = \frac{\rho a^2}{2 r \varepsilon_0}\hat{r}$ ✓

- 2c) (10 points) If the conductor is found to have a potential V_a with respect to some reference point, find the potential at all points inside and outside the distribution as a function of distance from the longitudinal symmetry axis

$V(r) = V(r_{ref}, r) = -\int_{r_{ref}}^{r} \vec{E}\cdot d\vec{r} = -\int_{r_{ref}}^{r} E\,dr$

$(r<a)$ $-\int_{r_{ref}}^{r} E\,dr = -\int_{r_{ref}}^{r} \frac{\rho r}{2\varepsilon_0}\,dr = -\frac{\rho}{2\varepsilon_0}\left(\frac{r^2}{2}\Big|_{r_{ref}}^{r}\right) = -\frac{\rho}{2\varepsilon_0}\left(\frac{r^2}{2} - \frac{r_{ref}^2}{2}\right)$

$= \frac{\rho}{\varepsilon_0}\left(r_{ref}^2 - r^2\right)$

let r_{ref} be at $r=a$. Then $(r<a)$ $V(r) = \frac{\rho}{\varepsilon_0}(a^2-r^2) + V_a$ and $V(r_{ref}) = 0$ × V_a

$(a<r<b)$ $\boxed{V(r) = 0} + V_a$

$(r>b)$ $V(r) = -\int_{a}^{r} \frac{\rho a^2}{2r\varepsilon_0}\,dr = -\frac{\rho a^2}{2\varepsilon_0}\int_{a}^{r}\frac{1}{r}\,dr = -\frac{\rho a^2}{2\varepsilon_0}\left(\ln\frac{r}{a}\right) = \boxed{\frac{\rho a^2}{2\varepsilon_0}\ln\left(\frac{a}{r}\right)}$ ×

~~7~~

6

3) A charged particle of mass m_1 and charge q_1 is shot directly towards a charge of mass m_2 and charge q_2 from an infinite distance away. If $q_1 q_2 > 0$, the initial speed of m_1 (measured in the frame in which m_2 is initially at rest) is v_0 and m_2 is free to move...

- 3a) (5 points) Assuming the system is correctly defined, which mechanical quantities are conserved and why?

momentum and ~~kinetic~~ total mechanical energy is conserved. This is because ~~electric potential energy~~ there are no external forces acting on the system. Electric potential energy + kinetic energy → total energy is conserved.

- 3b) (5 points) How fast is each particle moving when they get as close to one another as they can get?

3

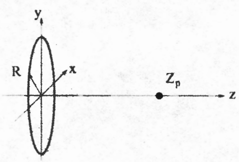
This happens when velocity of each particle is ~~zero~~ equal.

• 3c) (20 points) How close will the particles get to one another?

$PE_i + KE_i = PE_f + KE_f$

5

$$\frac{q_1 q_2}{4\pi\varepsilon_0 r_i} + \frac{1}{2} m_1 v_0^2 = \underbrace{\cancel{\frac{q_1 q_2}{4\pi\varepsilon_0 r_f}} \cancel{+0}}_{\frac{q_1 q_2}{4\pi\varepsilon_0 r_f} + \frac{1}{2}(m_1 + m_2) v_f^2}$$



- 1a) (5 points) If the circle shown represents a uniform **ring** of charge Q and radius R, what is the resultant electric field at z_p?

If you remember this, you don't need to derive it...

$$\boxed{\vec{E} = \frac{Qz}{4\pi\epsilon_0 (R^2+z_p^2)^{3/2}}\hat{k}}$$

- 1b) (5 points) If the circle shown represents a uniform disk of charge Q and radius R, what is the resultant electric field at z_p?

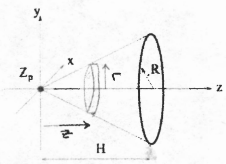
Again, if you remember this correctly (not as likely), you don't have to derive it. If you don't, build the thing out of infinitesimal rings... $R \to r$ $Q \to dq \to \sigma\, dA$

$$d\vec{E} = \frac{\sigma 2\pi r\, dr\, z}{4\pi\epsilon_0 (r^2+z_p^2)^{3/2}}\hat{k}$$

$$\int d\vec{E} = \frac{\sigma}{4\epsilon_0}\int_0^R \frac{2r\, dr}{(r^2+z_p^2)^{3/2}}\hat{k}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{1+R^2/z^2}}\right]\hat{k} \rightarrow \boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0 R^2}\left[1 - \frac{1}{\sqrt{1+R^2/z^2}}\right]\hat{k}}$$

$\sigma = \frac{Q}{\pi R^2}$



$$\frac{r}{z} = \frac{R}{H}$$

- 1c) (20 points) A charge distribution occupies the volume of a right circular cone of base-radius R and height H that is oriented so that its apex is on the origin and its longitudinal symmetry axis lies along the $+z$-axis with the base intersecting $z = H$, as shown. Assuming the charge distribution has a volume density given by

$$\rho(z) = \frac{(n+3)\,Q}{\pi R^2 H^{n+1}}\, z^n$$

find the (vector) electric field at the origin.

Build the cone from infinitesmal disks. $R \to r$ $Q \to dq \to \rho\, dv$

Small change in $\vec{E}$ field

$$\boxed{d\vec{E}} = \frac{\rho(z)\,\pi r^2 dz}{2\pi\epsilon_0 r^2}\left[1 - \frac{1}{\sqrt{1+(r/z)^2}}\right](-\hat{k})$$

we're on the left side of the distribution

dq — small change in charge

r & z are both variables so you need to express one in terms of the other one so you can integrate over just one variable

$$\int d\vec{E} = \frac{1}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{1+(R/H)^2}}\right]\int_0^H \rho(z)\, dz\ (-\hat{k})$$

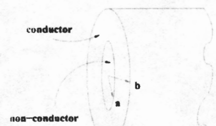
since $r = \frac{Rz}{H}$

i.e.

the infinitesimal change in volume of disks contains $d\vec{E}$ and dq

$$\boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0 R^2}\,\frac{n+3}{n+1}\left[1 - \frac{1}{\sqrt{1+(R/H)^2}}\right](-\hat{k})}$$

We're finding magnitudes here. Always take the domain of integration from a smaller to bigger value



2) An infinitely-long, non-conducting cylinder of radius a carries a uniform volume charge-density ρ. The non-conducting cylinder is, in turn, surrounded by a concentric, neutral conducting cylinder of inner-radius a and outer-radius b.

- 2a) (10 points) Find the amount of charge contained in a concentric cylinder of radius r, length L, for $r < a$, $a < r < b$ and $r > b$.

$(r<a)$	$q_{in}(r) = \rho\pi r^2 L$
$(a<r<b)$	$q_{in}(r) = 0$
$(b<r)$	$q_{in}(r) = \rho\pi a^2 L$

Gauss (cylindrical symmetry) $\vec{E} = \frac{q_{in}(r)/L}{2\pi\epsilon_0 r}\hat{r}$

$\vec{E} = 0$ inside the conductor

The conductor is neutral, all the induced charge is contained within the volume and so only the non-conducting cylinder contributes

- 2b) (10 points) Find the electric field (vector) at all points inside and outside the distribution, as a function of distance from the longitudinal symmetry axis.

Gauss' Law: Cylindrical Symmetry. $\vec{E} = \frac{q_{in}(r)/L}{2\pi\epsilon_0 r}\hat{r}$

$$\boxed{\vec{E} = \begin{cases} \frac{\rho}{2\epsilon_0} r\hat{r} & (r<a) \\ 0 & (a<r<b) \\ \frac{\rho a^2}{2\epsilon_0 r}\hat{r} & (b<r) \end{cases}}$$

- 2c) (10 points) If the conductor is found to have a potential V_a with respect to some reference point, find the potential at all points inside and outside the distribution as a function of distance from the longitudinal symmetry axis

$$\Delta V(\vec{r}_{ref}, \vec{r}) = -\int_{r_{ref}}^{r} \vec{E}\cdot d\vec{r}$$

$$V(\vec{r}) = V(\vec{r}_{ref}) - \int_{r_{ref}}^{r} E_r\, dr \quad \text{(radial field)}$$

$(r \le a)$ $V(r) = V_a - \int_a^r \frac{\rho r}{2\epsilon_0} dr$

$V(r) = V_a - \frac{\rho}{4\epsilon_0}(r^2 - a^2)$

$(a \le r \le b)$ $V(r) = V_a - \int_a^r 0\, dr$

$V(r) = V_a$

$(b \le r)$ $V(r) = V_a - \int_a^b 0\, dr - \int_b^r \frac{\rho a^2}{2\epsilon_0 r} dr$

$V(r) = V_a - \frac{\rho a^2}{2\epsilon_0}\ln(r/b)$

$$\boxed{V(r) = \begin{cases} V_a - \frac{\rho}{4\epsilon_0}(r^2 - a^2) & (r \le a) \\ V_a & (a \le r \le b) \\ V_a - \frac{\rho a^2}{2\epsilon_0}\ln\left(\frac{r}{b}\right) & (b \le r) \end{cases}}$$

3) A charged particle of mass m_1 and charge q_1 is shot directly towards a charge of mass m_2 and charge q_2 from an infinite distance away. If $q_1 q_2 > 0$, the initial speed of m_1 (measured in the frame in which m_2 is initially at rest) is v_0 and m_2 is free to move...

- 3a) (5 points) Assuming the system is correctly defined, which mechanical quantities are conserved and why?

⇒ "Correctly defined" = The system consists of m_1 & m_2

$\sum F_{ext} = 0$ So linear Momentum is Conserved

$W_{NC} = 0$ So Mechanical energy is Conserved

- 3b) (5 points) How fast is each particle moving when they get as close to one another as they can get?

At that instant, $\vec{V}_1 = \vec{V}_2$ (why? ¨)

$$\sum P_{xi} = \sum P_{xf}$$

$$m_1 V_0 = (m_1 + m_2) V_{1x} \Rightarrow V_{1x} = \frac{m_1}{m_1 + m_2} V_0$$

Both masses have a speed $V_1 = \frac{m_1}{m_1 + m_2} V_0$

- 3c) (20 points) How close will the particles get to one another?

$\Delta E = W_{NC} \Rightarrow E_i = E_f$

$$\frac{1}{2} m_1 V_0^2 = \frac{1}{2}(m_1 + m_2) V_1^2 + \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} - \frac{G m_1 m_2}{r_{12}}$$

$$\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} V_0^2 = \left(\frac{q_1 q_2}{4\pi\epsilon_0} - G m_1 m_2 \right) \frac{1}{r_{12}}$$

$$\boxed{r_{12} = \frac{2(m_1 + m_2)}{m_1 m_2 V_0^2} \left(\frac{q_1 q_2}{4\pi\epsilon_0} - G m_1 m_2 \right)}$$

$G \lll \frac{1}{4\pi\epsilon_0}$, and in all likelihood the masses aren't that large... It's probably ok to neglect gravity in this case...

It's also fun to note the appearance of reduced mass ($\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$).

Note also that the distance of closest approach is $\propto 1/V_0^2$ ~ does that seem reasonable?

↝ What happens when $\frac{q_1 q_2}{4\pi\epsilon_0} \Rightarrow G m_1 m_2$?? ¨