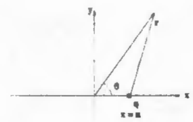


A point charge q sits on the $+x$ axis, a distance a from the origin. For each of the following parts, point P is located at a point described by the polar coordinates (r, θ) and all potentials are measured with respect to infinity.

- 1a) (5 points) Find the electric potential at point P.

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0\sqrt{r^2+a^2-2ar\cos\theta}}}$$

- 1b) (5 points) $(1-2bx+x^2)^{-1/2} \approx 1 + bx + \frac{1}{2}(3b^2-1)x^2 + \ldots$ when $x << 1$
  Expand the potential at point P in the limit $r >> a$ and comment on the result.

$$V(r) = \frac{q}{4\pi\epsilon_0 r}\left(1 - 2\cos\theta\,\frac{a}{r} + \frac{a^2}{r^2}\right)^{-1/2} \qquad b = \cos\theta$$

$$\boxed{V(r) \simeq \frac{q}{4\pi\epsilon_0 r}\left[1 + \frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}(3\cos^2\theta - 1)\ldots\right]}$$

V(r) has terms proportional to $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3} \ldots$
off the origin, the field due to a monopole
adds dipole, quadrupole... contributions to the field!

$b_i = \cos\theta_i$

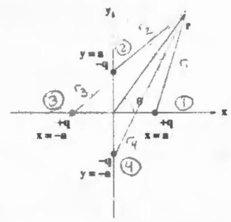
$b_1 = \cos\theta$

$b_2 = \cos(90^\circ-\theta) = \sin\theta$

$b_3 = \cos(180^\circ-\theta) = -\cos\theta$

$b_4 = \cos(90^\circ+\theta) = -\sin\theta$

$V_i = \frac{q_i}{4\pi\epsilon_0 r_i}$



The angles opposite r_i for each charge

$\theta_1 = \theta$

$\theta_2 = 90^\circ - \theta$

$\theta_3 = 90^\circ - \theta$

$\theta_4 = 90^\circ + \theta$

- 1c) (10 points) Find the electric potential at point P for the charge arrangement shown above. **Hint:** You do not have to start from scratch, but be real careful about identifying the relevant angle for each charge. Comment on the result.

use part b

$$V_1 = \frac{q}{4\pi\epsilon_0 r}\left[1 + \frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}(3\cos^2\theta - 1)\ldots\right]$$

$$V_2 = \frac{q}{4\pi\epsilon_0 r}\left[-1 - \frac{a}{r}\sin\theta - \frac{1}{2}\frac{a^2}{r^2}(3\sin^2\theta - 1)\ldots\right]$$

$$V_3 = \frac{q}{4\pi\epsilon_0 r}\left[1 - \frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}(3\cos^2\theta - 1)\ldots\right]$$

$$+ \quad V_4 = \frac{q}{4\pi\epsilon_0 r}\left[-1 + \frac{a}{r}\sin\theta - \frac{1}{2}\frac{a^2}{r^2}(3\sin^2\theta - 1)\ldots\right]$$

$$V_P = \frac{3qa^2}{4\pi\epsilon_0 r^3}(\cos^2\theta - \sin^2\theta)$$

$$\boxed{V_P = \frac{3a^2q}{4\pi\epsilon_0 r^3}\cos(2\theta)}$$

$V \propto \frac{1}{r^3} \Rightarrow$ this is a quadrupole!
(are you surprised? ;-)

- 1d) (10 points) Find the electric field (vector) at point P.

$$E_r = -\frac{\partial V}{\partial r} = \frac{9a^2q}{4\pi\epsilon_0 r^4}\cos(2\theta) \qquad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{6a^2q}{4\pi\epsilon_0 r^4}\sin(2\theta)$$

$$\boxed{\vec{E} = \frac{3a^2q}{4\pi\epsilon_0 r^4}\left[3\cos(2\theta)\,\hat{r} + 2\sin(2\theta)\,\hat{\theta}\right]}$$

A nonconducting sphere of radius a carries electric charge distributed with a volume charge-density $\rho(r) = \rho_0\left[1 - \frac{4}{3}\frac{r}{a}\right]$. It is surrounded by a concentric spherical conducting shell of inner-radius a and outer-radius b that carries an excess charge q.

$$q_{in}(r) = \int dq = \int \rho(r)\,dv = \int_0^r \rho(r)\,4\pi r^2\,dr$$

- 2a) (10 points) Find the amount of charge contained in a concentric sphere of radius r, for all values of r.

(r<a)

$$q_{in}(r) = \int_0^r \rho_0\left(1 - \tfrac{4}{3}\tfrac{r}{a}\right)4\pi r^2\,dr$$

$$q_{in}(r) = 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{3a}\right)$$

$$q_{in}(r) = \frac{4}{3}\pi\rho_0 r^3(1 - r/a)$$

(a<r<b)

The field inside the conductor is zero, Gauss' Law with spherical symmetry requires $q_{in}(r) \equiv 0$ *

* Guaranteed by induced charge

(b<r)

induced charge? $q_{in}(a) = 0 \Rightarrow Q_{induced} = 0$

all we have is the excess charge $q_{in}(r) = q$

$$\boxed{q_{in}(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3(1 - r/a) & (r<a) \\ 0 & (a<r<b) \\ q & (b<r) \end{cases}}$$

(Excess)

$Q = q_{in}(a)$ (induced)

- 2b) (10 points) Find the electric field (vector) at all points inside and outside the distribution.

Spherical Symmetry: $\vec{E} = \frac{q_{in}(r)}{4\pi\epsilon_0 r^2}\,\hat{r}$

$$\boxed{\vec{E} = \begin{cases} \frac{\rho_0 r}{3\epsilon_0}(1 - r/a)\,\hat{r} & r<a \\ 0 & a<r<b \\ \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r} & b<r \end{cases}}$$

- 2c) (10 points) Find the electric potential at all points inside and outside the distribution, if the potential at $r = a$ is defined to be V_a.

$$\Delta V(a,r) = V(r) - V(a) = -\int_a^r \vec{E}\cdot d\vec{r} = -\int_a^r E_r\,dr \qquad \text{(spherical symmetry)}$$

$$V(r) = V_a - \int_a^r E_r\,dr$$

(r<a)

$$V(r) = V_a - \frac{\rho_0}{3\epsilon_0}\int_a^r (r - r^2/a)\,dr$$

$$= V_a - \frac{\rho_0}{3\epsilon_0}\left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_a^r$$

$$= V_a - \frac{\rho_0}{3\epsilon_0}r^2\left[\frac{1}{2} - \frac{1}{3}\frac{r}{a} - \frac{1}{6}\frac{a^2}{r^2}\right]$$

$$= V_a - \frac{\rho_0 r^2}{18\epsilon_0}\left[3 - 2r/a - a^2/r^2\right]$$

(a<r<b)

$$V(r) = V_a - \int_a^r 0\,dr$$

$$V(r) = V_a$$

(b<r)

$$V(r) = V_a - \int_a^b 0\,dr - \int_b^r \frac{q}{4\pi\epsilon_0 r^2}\,dr$$

$$V(r) = V_a + \frac{q}{4\pi\epsilon_0 r}(1 - r/b)$$

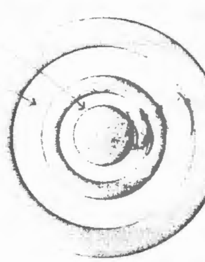
$$\boxed{V(r) = \begin{cases} V_a - \frac{\rho_0 r^2}{18\epsilon_0}\left(3 - 2r/a - a^2/r^2\right) & (r<a) \\ V_a & a<r<b \\ V_a + \frac{q}{4\pi\epsilon_0 r}(1 - r/b) & b<r \end{cases}}$$

$C_1 = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$

$C_2 = \frac{2\pi\epsilon_0 L}{\ln(e/d)}$

In Series:

$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$



A long solid conducting cylinder of radius a is surrounded by a long coaxial conducting cylindrical shell of inner-radius b and outer-radius d. That cylinder, in turn, is surrounded by a long coaxial conducting cylindrical shell of inner-radius e and outer-radius f.

- 1a) (10 points) How much capacitance will be measured (per unit length) between the inner-most and outer-most conductors?

Series:

$$C_{eq} = \frac{2\pi\epsilon_0 L \, \frac{1}{\ln(b/a)} \, \frac{1}{\ln(e/d)}}{\frac{1}{\ln(b/a)} + \frac{1}{\ln(e/d)}}$$

$$\boxed{\frac{C_{eq}}{L} = \frac{2\pi\epsilon_0}{\ln(b/a) + \ln(e/d)}}$$

- 1b) (5 points) Evaluate your answer to part a in the limit $b \to d$ and comment.

in the limit $b \to d$

$$\boxed{\frac{C_{eq}}{L} \to \frac{2\pi\epsilon_0}{\ln(e/a)}}$$

describes a simple cylindrical capacitor of inner radius a, outer radius e

✓

- 1c) (10 points) Assume the conductors start uncharged, and then, charge is transfered from the outer-most conductor to the inner-most conductor. What fraction of the total energy contained in the capacitor will be stored in the outer-most gap between the conductors?

$U = \frac{1}{2}\frac{Q^2}{C}$

$$\frac{U_2}{U_1+U_2} = \frac{1/C_2}{1/C_1 + 1/C_2}$$

$$\frac{U_2}{U_1+U_2} = \frac{1}{1+\frac{C_2}{C_1}}$$

$$\boxed{\frac{U_2}{U_1+U_2} = \frac{1}{1+\frac{\ln(b/a)}{\ln(e/d)}}}$$

- 1d) (5 points) Evaluate your answer to part b in the limit $a \to b$ and comment.

if $a \to b$, $\ln(b/a) \to 0$

$$\boxed{\frac{U_2}{U_1+U_2} \to 1}$$

There is no C_1 and 100% of the energy is stored in C_2

✓

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C = \frac{1}{\frac{C_2 + C_1}{C_1 C_2}} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C = \frac{1}{\frac{2\pi L\epsilon_0}{\ln(b/a)} + \frac{2\pi L\epsilon_0}{\ln(e/d)}}$$

$$\frac{(2\pi L\epsilon_0)^2 \frac{1}{\ln(b/a)\ln(e/d)}}{\frac{2\pi L\epsilon_0}{\ln(b/a)} + \frac{2\pi L\epsilon_0}{\ln(e/d)}} =$$

$$C = \frac{\ln(b/a) + \ln(e/d)}{2\pi L\epsilon_0}$$

$$\frac{\cancel{2\pi L\epsilon_0}\ \frac{1}{\cancel{\ln(b/a)}\ln(e/d)}}{\frac{\ln(b/a) + \ln(e/d)}{\cancel{\ln(b/a)}\ln(e/d)}}$$

$$= \frac{2\pi L\epsilon_0}{\ln(b/a) + \ln(e/d)}$$

$b \to d$

$$\ln\left(\frac{d}{a}\right) + \ln\left(\frac{e}{d}\right)$$

$$\cancel{\ln d} - \ln a + \ln e - \cancel{\ln d}$$

$$\ln\left(\frac{e}{a}\right)$$

$$\frac{2\pi L\epsilon_0}{\ln(e/a)}$$