3) The picture above shows an isolated conducting sphere of radius $2a$, far-removed from Earth-ground and a spherical capacitor of inner radius a and outer-radius $2a$. The isolated sphere has an initial potential V with respect to Earth-ground. The outer-conductor of the spherical capacitor is connected to Earth-ground by a thin, ideal conductor.

- 3a) (5 points) How much charge does the isolated sphere carry?

$V = \frac{kQ}{r}$ $\quad Q = \frac{Vr}{k}$

$Q = 4\pi\epsilon_0 \cdot 2a\, V$

$Q = 8\pi\epsilon_0 a \cdot V$

5

- 3b) (5 points) If a thin ideal conductor is connected from the isolated sphere to the inner-conductor of the spherical capacitor, how much charge will be present on each of the conducting surfaces in the problem?

sphere: $Q = 8\pi\epsilon_0 a V$ $\qquad C = \frac{Q}{\Delta V} = \frac{4\pi\epsilon_0 ab}{b-a}$

capacitor: $Q = -8\pi\epsilon_0 a V$

2

- 3c) (5 points) By how much did the energy of the system change when the new conductor was added? Explain.

$\Delta U = \frac{1}{2}\Delta Q \Delta V$

$= \frac{1}{2}\left([Q_f \Delta V] - [Q_i \Delta V]\right)$

$= \frac{1}{2}\left(16\pi\epsilon_0 a V^2\right)$

$= 8\pi\epsilon_0 a V^2$

1

$C = 4\pi\epsilon_0 \frac{ab}{b-a}$

- 3d) (5 points) Disconnect the wire joining the two spheres. How much work would have to be done to completely fill the spherical capacitor with a dielectric of constant κ?

$$W = \Delta U = \left[K 4\pi\epsilon_0 \frac{ab}{b-a}\right] - \left[4\pi\epsilon_0 \frac{ab}{b-a}\right]$$

$$= 4\pi\epsilon_0 [K-1] \left[\frac{2a^2}{2a-a}\right]$$

$$\boxed{W = 8\pi\epsilon_0 a [K-1]}$$

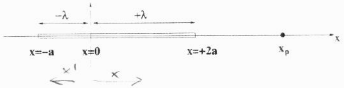
- 3e) (10 points) Having filled the spherical capacitor with dielectric, the wire that previously connected the spheres is re-connected. How much charge will travel down the wire? Assuming positive charge-carriers, in which direction will that charge travel?

$$C = K 4\pi\epsilon_0 \frac{ab}{b-a} = 8K\pi\epsilon_0 a$$

$$Q = CV$$

$$\boxed{= 8K\pi\epsilon_0 a \cdot V}$$

It will travel from capacitor to sphere



1) The sketch above shows an electrically-charged rod. One portion of the rod extends from $x = -a$ to $x = 0$ and has a linear charge density $-\lambda$. The remainder of the rod extends from $x = 0$ to $x = +2a$ and has a linear charge density $+\lambda$.

- 1a) (10 points) Find the electric potential (with respect to infinity) at the point x_p, where $x_p > 2a$.

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \qquad dq_1 = -\lambda\, dx' \qquad dq_2 = +\lambda\, dx$$

$$V = \frac{1}{4\pi\epsilon_0}\int_0^{-a} \frac{-\lambda\, dx'}{x_p + x'} + \frac{1}{4\pi\epsilon_0}\int_0^{2a} \frac{\lambda\, dx}{x_p - x}$$

$$V = \frac{-\lambda}{4\pi\epsilon_0}\left[\ln(x_p + x')\right]_0^{-a} + \frac{\lambda}{4\pi\epsilon_0}\left[-\ln(x_p - x)\right]_0^{2a}$$

$$V = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\frac{x_p}{x_p - 2a}\right] - \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{x_p - a}{x_p}\right)\right]$$

$$V = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{x_p}{x_p - 2a}\right) - \ln\left(\frac{x_p - a}{x_p}\right)\right] \quad \checkmark\ /10$$

$$\cancel{V = \frac{\lambda}{4\pi\epsilon_0}\ln[}$$

- 1b) (10 points) Find the (vector) electric field for every point $x > 2a$ along the positive x-axis. Be clear how you obtain each component.

$E_y = E_z = 0$ symmetry

$$E_x = \frac{kq}{r^2}$$

$$E_x = \frac{1}{4\pi\epsilon_0}\int_0^{-a}\frac{-\lambda\, dx'}{(x_p + x')^2} + \int_0^{2a}\frac{\lambda\, dx}{(x_p - x)^2}$$

$$E_x = \nabla V = \frac{dV_x}{dx}$$

$$E_x = \frac{d\left[\frac{\lambda}{4\pi\epsilon_0}\left[\ln\frac{x_p}{x_p - 2a} - \ln\frac{(x_p - a)}{x_p}\right]\right]}{dx}$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{x_p}{x_p - 2a} - \frac{x_p - a}{x_p}\right] \quad \checkmark\ /0$$

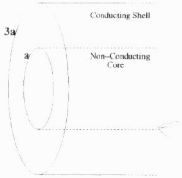
- 1c) (5 points) Evaluate the electric field in the limit $x >> 2a$, and explain the result.

$$\underline{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{x_p}\left[0 - \frac{2a - a}{2a}\right]$$

$$\underline{E} = \frac{\lambda}{4\pi\epsilon_0 a}\, x_p$$

It should be like a point charge of $+\frac{\lambda}{a}$

- 1d) (5 points) Find the monopole and dipole moments of the charge distribution. (**Hint:** $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$)



2) A long cylindrical, non-conducting core of radius a, and volume charge density $\rho(r) = \frac{\rho_0}{a^2}\, r\,(a-r)$ is surrounded by a neutral cylindrical conducting shell of inner-radius a and outer-radius $3a$.

- 2a) (10 points) Find the total charge enclosed by a fictitious co-axial cylinder of length L and radius r. Do this for all values of r. (Careful!)

$r < a : Q = 0$ $\quad V = \pi r^2 L$ $\quad \rho_0 : \frac{Q}{\ }$

$a \le r < 3a$: $Q = \int \rho \, dV$ $\quad dv = 2\pi r L dr$

$Q = \int_a^{3a} \frac{\rho_0}{a^2} r(a-r)\; 2\pi r L \, dr$

$Q = \int_a^{3a} \frac{\rho_0 2\pi L}{a^2}\; r^2 a - r^3 dr$ $\quad > \quad Q = \frac{\rho_0 2\pi L}{a^2} r(a-r) \int_a^{3a} r dr$

$Q = \frac{\rho_0 2\pi L}{a^2} \left[a\frac{r^3}{3} - \frac{r^4}{4} \right]_a^{3a}$ $\quad Q = \rho \cdot \left[\frac{r^2}{2}\right]_a^{3a}$

2

$Q = \frac{\rho_0 2\pi L}{a^2} \left[a \cdot \frac{9a^3}{3} - \frac{27a^4}{4} - \frac{a^4}{3} + \frac{a^4}{4} \right]$ $\quad Q = \rho \cdot \frac{9a^2 - a^2}{2}]$

$Q = \frac{\rho_0 2\pi L}{a^2} \left[\frac{8a^4}{3} - \frac{28a^4}{4} \right]$ $\quad Q = \rho \cdot 4a^2$

$\quad Q = \frac{\rho_0}{a^2} r(a-r) 4a^2$

$Q = \rho_0 2\pi L\, a^2 \left[\frac{8}{3} - 7 \right]$ $\quad Q = 4\rho_0 r(a-r) \cdot 2\pi L$

$\quad Q = 8\pi L \rho_0 r(a-r)$

$r \ge 3a : Q = \rho V = \frac{\rho_0}{a^2} r(a-r) \pi r^2 L$

$Q = \frac{\rho_0}{a^2} \left[a\pi r^2 - \pi r^3 \right]$

- 2b) (10 points) Find the (vector) electric field as a function of radial distance from the axis of the cylinders, for all values of that radial distance.

$E = \frac{kQ}{r^2}$

~~$Q = 4\rho_0 \pi a r^2$~~ $Q = 8\pi L\rho_0[r(a-r)$

~~$dq = 4\rho_0 a \cdot 2r\,dr$~~ $dq = 8\pi L\rho_0[a - 2r]$

$r < a: \quad E = 0$

$a < r < 3a: E = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}$

$= \frac{1}{4\pi\epsilon_0}\int_a^{3a} \frac{8\pi L\rho_0[a-2r]\,dr}{r^2}$

$= \frac{2L\rho_0}{\epsilon_0}\int_a^{3a} \frac{a-2r}{r^2}\,dr \quad u = r^2 \quad du = 2r\,dr$

1

$= \frac{2L\rho_0}{\epsilon_0}\int_a^{3a} \frac{a}{r^2} - \int_{a^2}^{9a^2} u^{-1}\,du$

$= \frac{2L\rho_0}{\epsilon_0}\left[-\frac{a}{r}\right]_a^{3a} - \ln\left[\frac{9a^2}{a^2}\right] = \frac{2L\rho_0}{\epsilon_0}\left[1 - \frac{1}{3}\right] - \ln 9) = \frac{2\pi\rho_0}{\epsilon_0}\left(\frac{2}{3}\cdot\right)$

for $a > 3a$:

$E = \frac{kQ}{r}$

$E = \frac{8\pi L\rho_0[r(a-r)]}{4\pi\epsilon_0 r}$

$E = \frac{2L\rho(a-r)}{\epsilon_0}$

- 2c) (10 points) Suppose a particle of mass m and charge q were released at rest from a point very near the axis of the cylinders. If it is observed to shoot outward from the axis, i) what is the sign of the test charge and ii) how fast is it moving as it leaves the outer-radius of the conductor?

i. Negative

1

ii. $\frac{1}{2}mv^2 = Eq$

$v^2 = \frac{2E(3a)q}{m}$

$v^2 = 2 \cdot \frac{2L\rho(a-3a)}{\epsilon_0}$

$v^2 = \frac{4L\rho(-2a)}{\epsilon_0}$

$v = 2\sqrt{\frac{2L\rho a}{\epsilon_0}}$