MT1 Physics 1B-5, S15

Full Name (Printed) ______________________

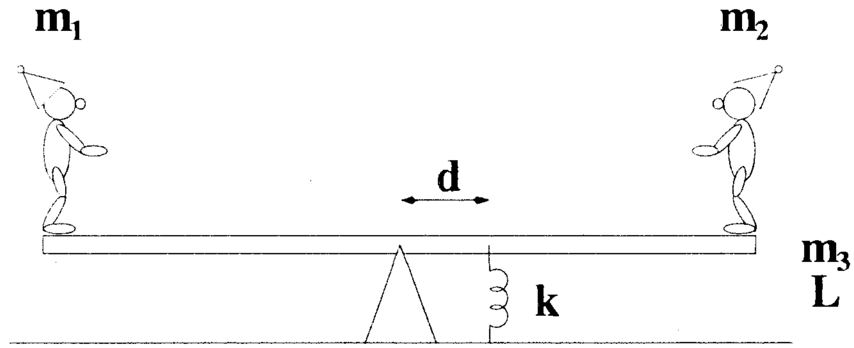
Full Name (Signature)______________________

Student ID Number ______________________

Seat Number ______________________

Problem	Grade
1	19 /30
2	11 /30
3	20 /30
Total	50 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**

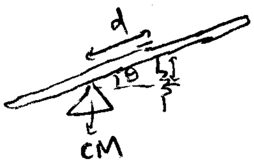


Consider the child's toy shown above. A uniform rod, of mass m_3 and length L is attached to a pivot that passes through its center of mass. Small clowns, of mass m_1 and m_2 are glued on opposite ends of the rod. A spring, of constant k, is mounted vertically between the base of the toy and the rod, a distance d from the rod's center of mass. The spring is not extended or compressed when the rod is horizontal. The rod is displaced slightly, and the system begins to rock...

- 1a) (15 points) Obtain the differential equation that describes the motion of the system for small displacements from equilibrium. (**Hint:** for a uniform stick, $I_{cm} = \frac{1}{12}ML^2$)

known: m_1, m_2, m_3, L, k, d

$\Sigma\tau = I\alpha$



$$-|kd\sin\theta||d\cos\theta| - |m_2 g||\tfrac{L}{2}\cos\theta| = I\frac{d^2\theta}{dt^2}$$

$$-kd^2\sin\theta\cos\theta - \frac{m_2 gL\cos\theta}{2} = I\frac{d^2\theta}{dt^2} = \left(\frac{1}{12}M_3L^2 + m_2\frac{L^2}{4} + m_1\frac{L^2}{4}\right)\frac{d^2\theta}{dt^2}$$

$$I\frac{d^2\theta}{dt^2} + kd^2\sin\theta\cos\theta + \frac{m_2 gL\cos\theta}{2} \underbrace{-\ m_1 gL\cos\theta}_{} = 0$$

For small θ, $\sin\theta \approx \theta$, $\cos\theta \approx 1$

$$\therefore I\frac{d^2\theta}{dt^2} + kd^2\theta + \frac{m_2 gL}{2} \approx 0$$

$$\cancel{\frac{d^2\theta}{dt^2} + \frac{12kd^2}{M_3L^2}\theta + \frac{6m_2 g}{m_3 L} \approx 0}$$

$$\frac{d^2\theta}{dt^2} + \frac{kd^2}{I}\theta + \frac{m_2 gL}{2I} = 0, \text{ where } I = \frac{1}{12}m_3L^2 + m_2\frac{L^2}{4} + m_1\frac{L^2}{4}$$

13/15

✓

- 1b) (5 points) At what angular frequency will the system oscillate? What is the value of θ when the system is in equilibrium?

$$\omega = \sqrt{\frac{kd^2}{I}} = \sqrt{\frac{kd^2}{\frac{m_3L^2}{12} + \frac{m_2L^2}{4} + \frac{m_1L^2}{4}}} = \sqrt{\frac{kd^2}{\frac{L^2}{4}\left(\frac{1}{3}m_3 + m_2 + m_1\right)}}$$

$$\boxed{= \frac{2}{L}\sqrt{\frac{kd^2}{\frac{m_3}{3} + m_2 + m_1}}}$$

$$\text{Set } \frac{d^2\theta}{dt^2} = 0 \Rightarrow \frac{kd^2}{I}\theta = -\frac{m_2 gL}{2I} \Rightarrow kd^2\theta = -\frac{m_2 gL}{2}$$

$$\boxed{\theta = -\frac{m_2 gL}{2kd^2}}$$

5/5

- 1c) (10 points) Suppose, once you set the system in motion, that it really oscillates at a frequency $\omega = F\omega_0$, where ω_0 is the natural frequency of the system (and F is some dimensionless constant, close, but not equal to 1). How many complete cycles will the toy make before its **energy** falls to e^{-3} of it's original value?

$$\theta = \theta_0 e^{-\frac{bt}{2m}}\cos(\omega t) \qquad E = E_0 e^{-\frac{bt}{2m}}$$

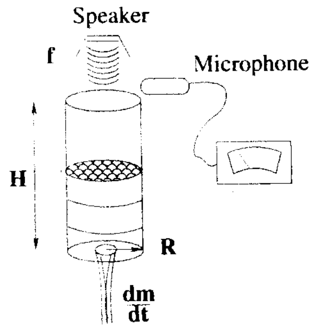
$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

1/10

$$\text{Energy} = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2_{max}$$

$$\frac{d\theta}{dt} = -\omega\theta_0 e^{-\frac{bt}{2m}}\sin(\omega t) - \frac{b}{2m}\theta_0 e^{-\frac{bt}{2m}}\cos(\omega t)$$

$$\left(\frac{d\theta}{dt}\right)_{max} \text{ occurs when } \omega t = \frac{\pi}{2} \Rightarrow \left(\frac{d\theta}{dt}\right)_{max} = \omega\theta_0 e^{-\frac{b\pi}{m}}$$



Consider the apparatus shown above... Sound (of frequency f) is emitted by a speaker into a tube of radius R and height H. A microphone, placed near the open end of the tube, is used to monitor the intensity of the sound that is re-radiated from the tube. There is liquid (of volume mass density ρ) partially filling the tube, and it is leaking from a hole (of negligible area) in the bottom of the tube at a rate $\frac{dm}{dt}$ (m is mass).

- 2a) (10 points) What is the lowest frequency that will resonate in the tube? What are the boundary conditions at resonance? The more (correct) details you can give, the more points you will get.

known: R, H, f Lowest frequency occurs at the longest wavelength. This is when the tube is empty.

$\therefore \lambda = 4H \Rightarrow f = \frac{V_s}{\lambda} = \frac{V_s}{4H}$ where V_s is velocity of sound.

The boundary conditions of resonance are a node at the bottom of the tube and an anti-node at the open end. This means amplitude is zero at the bottom while amplitude is maximum at the open end.

10

- 2b) (10 points) Assuming the speed of sound in air is v_{snd}, how frequently will the microphone record intensity maxima as the water leaks out? [Hints: Under what condition will the re-radiated sound be at maximum intensity? How frequently does this condition occur? Call this frequency f_{peak} to avoid confusion with f].

$V_{snd} = f\lambda$

Intensity maxima occurs when $x = (2N+1)\frac{\lambda}{4}$, where $N = 0, 1, 2, \ldots$ and x is the distance from open end of the tube to the surface of the liquid.

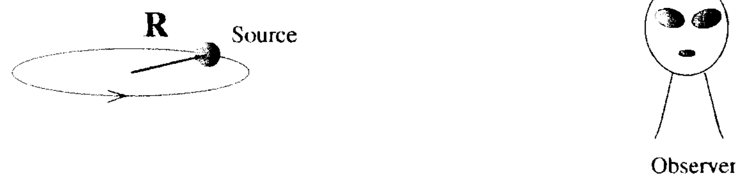
$\therefore \lambda = \frac{4x}{2N+1}$

$f = (2n+1) f_0$ where $n = 0, 1, 2, \ldots$

$f_0 = \frac{V_{snd}}{4L}$

- 2c) (10 points) If you're creative, you can use a device like this to measure the speed of sound in air. What parameter would you vary? What parameter would you record? Make a qualitative plot of one parameter vs. the other - and find v_{snd} as a function of the properties of that plot (for instance, if the plot is linear, How would you obtain v_{snd} from the measured slope and intercept?)

0



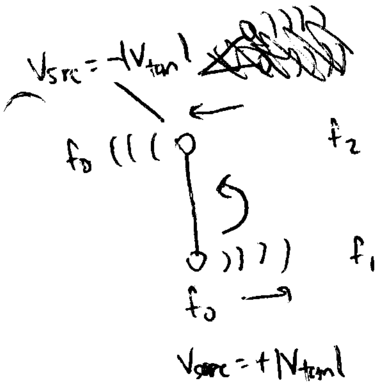
3) If a source that emits sound at a single frequency is tied to a string of length R and twirled in a horizontal circle as shown, an observer will hear a distribution of frequencies, characterized by a relative width

$$\sigma = \frac{\Delta f}{f_{avg}} = \frac{2(f_{hi} - f_{lo})}{f_{hi} + f_{lo}}$$

where f_{hi} and f_{lo} are the highest and lowest frequencies heard, respectively.

- 3a) (20 points) Given σ, R and V_{snd}, what is the angular velocity of the source?

Find ω.



~~f_{high}~~ f_{hi} occurs when the source is moving at maximum velocity towards observer.
f_{lo} occurs when the source is moving at maximum velocity away from observer.

$$\frac{f_1}{f_0} = \frac{V_{snd}}{V_{snd} - |V_{tan}|} \quad \checkmark \quad 5$$

$$\frac{f_2}{f_0} = \frac{V_{snd}}{V_{snd} + |V_{tan}|} \quad \checkmark \quad 5$$

$f_1 = f_{hi}$ while $f_2 = f_{lo}$

$$f_1 - f_2 = f_0\left(\frac{f_1}{f_0} - \frac{f_2}{f_0}\right) = f_0\left(\frac{V_{snd}}{V_{snd} - V_{tan}} - \frac{V_{snd}}{V_{snd} + V_{tan}}\right) = f_0\left(\frac{V_{snd}(V_{snd} + V_{tan} - 1)}{V_{snd}^2 - V_{tan}^2}\right)$$

$$f_1 + f_2 = f_0\left(\frac{f_1}{f_0} + \frac{f_2}{f_0}\right) = f_0\left(\frac{V_{snd}}{V_{snd} - V_{tan}} - \frac{V_{snd}}{V_{snd} + V_{tan}}\right) = f_0\left(\frac{V_{snd}(V_{snd} + V_{tan} - 1)}{V_{snd}^2 - V_{tan}^2}\right)$$

3

$$\sigma = \frac{2(f_1 - f_2)}{f_1 + f_2} = \frac{2\left(\frac{f_1}{f_0} - \frac{f_2}{f_0}\right)}{\frac{f_1}{f_0} + \frac{f_2}{f_0}} = \frac{2\left[\frac{V_{snd}(V_{snd} + V_{tan} - 1)}{V_{snd}^2 - V_{tan}^2}\right]}{\left[\frac{V_{snd}(V_{snd} + V_{tan} - 1)}{V_{snd}^2 - V_{tan}^2}\right]} = 2$$

X

$V_{tan} = r\omega$ ✓ 3

16

• 3b) (5 points) What is the actual frequency emitted by the source?

$$\frac{f_{obs}}{\cancel{f_{snd}}\ f_{src}} = \frac{V_{snd}-V_{obs}}{V_{snd}-V_{src}}$$

$$f_{src} = \frac{(V_{snd}-V_{src})f_{obs}}{V_{snd}-V_{obs}}$$ ✓

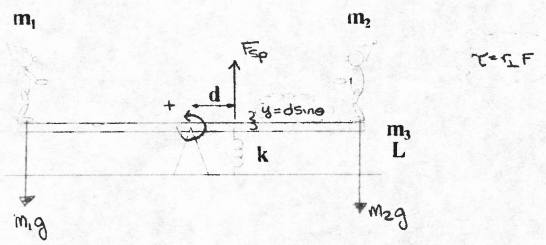
2

• 3c) (5 points) An astronomer passes the light from a distant star through a prism to observe the spectral lines that tell her what the chemical makeup of the star is. She notes that the lines, which are normally rather sharp and well-defined when the source is in the laboratory, are rather fuzzy. Given that a) light is a wave and b) prisms spread light by wavelength, explain how the astronomer might use the observation to determine the rotational rate of the star. [This is how it's done in real-life!]

She observes the change in colour of the star to determine the rotational rate. Since different colors of light have different frequencies, different observed frequencies will correspond to diff. colors. The rate of change of colors can determine the rotational rate.

x /

2



Consider the child's toy shown above. A uniform rod, of mass m_3 and length L is attached to a pivot that passes through its center of mass. Small clowns, of mass m_1 and m_2 are glued on opposite ends of the rod. A spring, of constant k, is mounted vertically between the base of the toy and the rod, a distance d from the rod's center of mass. The spring is not extended or compressed when the rod is horizontal. The rod is displaced slightly, and the system begins to rock...

- 1a) (15 points) Obtain the differential equation that describes the motion of the system for small displacements from equilibrium. (**Hint:** for a uniform stick, $I_{cm} = \frac{1}{12}ML^2$)

$$\tau_1 = \tfrac{1}{2} m_1 g L \cos\theta \qquad \tau_1 \approx \tfrac{1}{2} m_1 g L$$
$$\tau_2 = -\tfrac{1}{2} m_2 g L \cos\theta \quad \xrightarrow{\text{Small angle}} \quad \tau_2 \approx -\tfrac{1}{2} m_2 g L$$
$$\tau_3 = -kd^2 \sin\theta \cos\theta \qquad \tau_3 \approx -kd^2\theta$$

← linear restoring torque

$$\Sigma \tau = I\alpha$$

$$-\tfrac{1}{2}(m_2 - m_1) g L - kd^2\theta = \tfrac{1}{12} m_3 L^2 \frac{d^2\theta}{dt^2}$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{12kd^2}{m_3 L^2}\theta = \frac{6(m_1 - m_2) g}{m_3 L}}$$

- 1b) (5 points) At what angular frequency will the system oscillate? What is the value of θ when the system is in equilibrium?

Reading from the differential equation...

$$\boxed{\omega_0 = \frac{2d}{L}\sqrt{\frac{3k}{m_3}} \qquad \theta_{eq} = \frac{(m_1 - m_2) g L}{2kd^2}}$$

→ set $\frac{d^2\theta}{dt^2} = 0$

- 1c) (10 points) Suppose, once you set the system in motion, that it really oscillates at a frequency $\omega = F\omega_0$, where ω_0 is the natural frequency of the system (and F is some dimensionless constant, close, but not equal to 1). How many complete cycles will the toy make before its **energy** falls to e^{-3} of it's original value?

we'll write things in terms of b & m but keep in mind, they're just place-holders

$$E = E_0 e^{-\frac{bt}{m}}$$

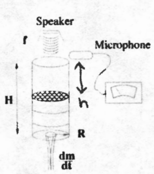
$$T = \frac{2\pi}{\omega} \qquad T = \frac{2\pi}{\sqrt{\omega_0^2 - (\frac{b}{2m})^2}}$$

$$\omega^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$$
$$F^2\omega_0^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$$
$$\left(\frac{b}{2m}\right)^2 = \omega_0^2(1 - F^2)$$
$$\left(\frac{b}{2m\omega_0}\right)^2 = 1 - F^2$$

$$3 = \frac{b}{m}(NT)$$
$$\frac{3}{2} = \frac{b}{2m}(NT)$$
$$\frac{3}{2} = N \frac{2\pi}{\sqrt{(\frac{2m\omega_0}{b})^2 - 1}}$$
$$N = \frac{3}{4\pi}\sqrt{\left(\frac{2m\omega_0}{b}\right)^2 - 1}$$
$$N = \frac{3}{4\pi}\sqrt{\frac{1}{1-F^2} - 1}$$

$$\boxed{N = \frac{3}{4\pi}\frac{F}{\sqrt{1 - F^2}}}$$

what happens when $F \to 1$? why?



Consider the apparatus shown above... Sound (of frequency f) is emitted by a speaker into a tube of radius R and height H. A microphone, placed near the open end of the tube, is used to monitor the intensity of the sound that is re-radiated from the tube. There is liquid (of volume mass density ρ) partially filling the tube, and it is leaking from a hole (of negligible area) in the bottom of the tube at a rate $\frac{dm}{dt}$ (m is mass).

- 2a) (10 points) What is the lowest frequency that will resonate in the tube? What are the boundary conditions at resonance? The more (correct) details you can give, the more points you will get.

In resonance, there will be an antinode at the open end (top) and a node at the closed end (bottom) if we're looking at displacement. It's the opposite if we're looking at pressure. The lowest fundamental occurs when the tube is empty...

$$\boxed{f_0 = \frac{V_{snd}}{4H}}$$

- 2b) (10 points) Assuming the speed of sound in air is v_{snd}, how frequently will the microphone record intensity maxima as the water leaks out? [Hints: Under what condition will the re-radiated sound be at maximum intensity? How frequently does this condition occur? Call this frequency f_{peak} to avoid confusion with f].

Resonance occurs when $f = (2N+1)\dfrac{V_{snd}}{4h_n}$

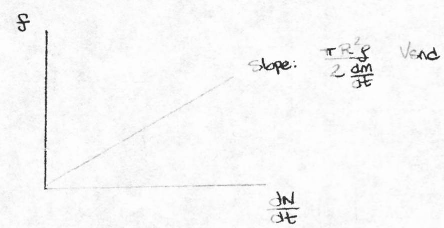
$$\frac{dh_n}{dt} = \frac{dN}{dt}\frac{V_{snd}}{2f}$$

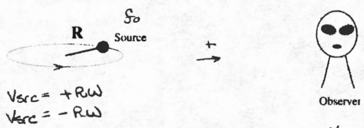
$$Vol = \pi R^2 h \qquad M = \rho(Vol)$$
$$\frac{d(Vol)}{dt} = \pi R^2 \frac{dh}{dt} \qquad \frac{d(Vol)}{dt} = \frac{1}{\rho}\frac{dM}{dt}$$
$$\frac{dh}{dt} = \frac{1}{\pi R^2 \rho}\frac{dm}{dt}$$

$$\boxed{\frac{dN}{dt} = \frac{2f}{V_{snd}}\frac{1}{\pi R^2 \rho}\frac{dm}{dt}}$$

- 2c) (10 points) If you're creative, you can use a device like this to measure the speed of sound in air. What parameter would you vary? What parameter would you record? Make a qualitative plot of one parameter vs. the other - and find v_{snd} as a function of the properties of that plot (for instance, if the plot is linear, How would you obtain v_{snd} from the measured slope and intercept?)

It's not going to be particularly trivial to alter ρ, R or $\frac{dm}{dt}$. Ok, well, I suppose we could change fluids, but we'd have to recalibrate each time. Probably a good second-order pass... How about...





app: $V_{src} = +R\omega$
dep: $V_{src} = -R\omega$

$V_{obs} = 0$

3) If a source that emits sound at a single frequency is tied to a string of length R and twirled in a horizontal circle as shown, an observer will hear a distribution of frequencies, characterized by a relative width

$$\sigma = \frac{\Delta f}{f_{avg}} = \frac{2(f_{hi} - f_{lo})}{f_{hi} + f_{lo}}$$

$$\frac{f_{obs}}{f_{emit}} = \frac{V_{snd} - V_{obs}}{V_{snd} - V_{src}}$$

where f_{hi} and f_{lo} are the highest and lowest frequencies heard, respectively.

- 3a) (20 points) Given σ, R and V_{snd}, what is the angular velocity of the source?

$$\frac{f_{hi}}{f_0} = \frac{V_{snd}}{V_{snd} - R\omega} \qquad \frac{f_{lo}}{f_0} = \frac{V_{snd}}{V_{snd} + R\omega}$$

$$f_0\left(\frac{1}{f_{lo}} + \frac{1}{f_{hi}}\right) = 2$$

$$f_0\left(\frac{f_{lo} + f_{hi}}{f_{lo}\, f_{hi}}\right) = 2$$

$$f_0\left(\frac{1}{f_{lo}} - \frac{1}{f_{hi}}\right) = \frac{2R\omega}{V_{snd}}$$

$$f_0\left(\frac{f_{hi} - f_{lo}}{f_{lo}\, f_{hi}}\right) = \frac{2R\omega}{V_{snd}}$$

$$\frac{f_{hi} - f_{lo}}{f_{hi} + f_{lo}} = \frac{\sigma}{2} = \frac{R\omega}{V_{snd}} \quad \Rightarrow \quad \boxed{\omega = \frac{\sigma}{2}\frac{V_{snd}}{R}}$$

- 3b) (5 points) What is the actual frequency emitted by the source?

Probably easiest to use f_{lo} and f_{hi}... then

$$\boxed{f_0 = \frac{2 f_{lo} f_{hi}}{f_{lo} + f_{hi}}}$$

- 3c) (5 points) An astronomer passes the light from a distant star through a prism to observe the spectral lines that tell her what the chemical makeup of the star is. She notes that the lines, which are normally rather sharp and well-defined when the source is in the laboratory, are rather fuzzy. Given that a) light is a wave and b) prisms spread light by wavelength, explain how the astronomer might use the observation to determine the rotational rate of the star. [This is how it's done in real-life!]

⇒ Though the equation for Doppler is a little different when you're talking about light, the basic idea is the same...

If we assume $\omega \propto \sigma$, we can estimate the rotational rate of the star by examining the relative width of the spectral line!