MT1 Physics 1B, S13

Full Name (Printed) ______________________

Full Name (Signature) ______________________

Student ID Number ______________________

Seat Number ______________________

Problem	Grade
1	15 /30
2	25 /30
3	26 /30
Total	66 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**

1) When an object is immersed (fully or partly) in a fluid (or gas), it experiences an upward 'buoyant' force equal to the weight of the fluid (gas) displaced by the object.

For the following questions, when relevant, use the equation $\rho_{air}(y) = \rho_0 e^{-y/H}$, to model the volume mass density of the Earth's atmosphere, where ρ_0 and H are characteristic constants and y is the height above the Earth's surface.

- 1a) (5 points) Suppose a small hot-air balloon (of volume mass density ρ) is released into the atmosphere. Write the (exact) differential equation that describes the motion of the balloon.

let V be volume of balloon and volume of air displaced

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

$\Sigma F = ma = \rho_0 e^{-y/H} Vg - \rho Vg$

$m\frac{d^2y}{dt^2} = Vg(\rho_0 e^{-y/H} - \rho)$

$m = \rho V$

$$\boxed{\frac{d^2y}{dt^2} = \frac{Vg}{\cancel{\rho V}}\left(\rho_0 e^{-y/H} - \cancel{\rho}\right)}$$

$\frac{g}{\rho}$ $\frac{\rho_0}{\rho}$

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- 1b) (5 points) At what height will the balloon be in equilibrium?

equilibrium $\Sigma F = 0 = \rho_0 e^{-y/H} Vg - \rho Vg$

$\rho_0 e^{-y/H} = \rho$

$\frac{\rho}{\rho_0} = e^{-y/H}$

5

$$\boxed{-H \ln\left(\frac{\rho}{\rho_0}\right) = y}$$

✓

• 1c) (10 points) Define $\delta y = y - y_{eq}$, where y_{eq} is the relation you found in part b. Execute a change of variables in which you re-write the differential equation from part b in terms of δy. (Hint: the second derivative of a constant is zero).

0

let $\delta y = y - y_{eq}$

$\delta y'' = y''$

$$\frac{d^2 \delta y}{dt^2} = \frac{Vg}{m}\left(\rho_0 e^{-\delta y/H} - \rho\right)$$

$$\delta y = y + H \ln\left(\frac{\rho}{\rho_0}\right)$$

• 1d) (10 points) Suppose δy is rather small compared to H. Show that the balloon will oscillate around equilibrium and find the angular frequency of that oscillation.

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if δy is small, can approx with taylor series: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

use first 2 terms.

so $e^{-\delta y/H} = 1 + \left(-\frac{\delta y}{H}\right)$

$$\frac{d^2 \delta y}{dt^2} = \frac{Vg}{m}\left(\rho_0\left(1 - \frac{\delta y}{H}\right) - \rho\right)$$

$$\frac{d^2 \delta y}{dt^2} + \frac{Vg\rho_0}{m}\left(\frac{\delta y}{H} - 1\right) = -\frac{Vg\rho}{m}$$

let $u = \frac{\delta y}{H} - 1$

$\frac{d^2 u}{dt^2} = \frac{d^2 \delta y}{dt^2} \cdot \frac{1}{H}$

$$H\frac{d^2 u}{dt^2} + \frac{Vg\rho_0}{m}(u) = -\frac{Vg\rho}{m}$$

$m = \rho V$

$$\frac{d^2 u}{dt^2} + \frac{g\rho_0}{\rho H} u = -g$$

$$\frac{d^2 u}{dt^2} + g\left(\frac{\rho_0}{\rho H} u - 1\right) = 0$$

$$\frac{d^2 u}{dt^2} + g\frac{\rho_0}{\rho H}\left(u - \frac{\rho H}{\rho_0}\right) = 0$$

let $v = u - \frac{\rho H}{\rho_0}$

$\frac{d^2 v}{dt^2} = \frac{d^2 u}{dt^2}$

$$\frac{d^2 v}{dt^2} + \frac{g\rho_0}{\rho H} v = 0$$

(SHM)

$$\omega = \sqrt{\frac{g\rho_0}{\rho H}}$$

2) The spring and shock-absorber combination attached to the front wheel of a motorcyle constitutes a damped oscillating system. If you were to push down on the handlebars of a cycle and release them, you might observe vertical oscillations that could be described by the differential equation:

$$\frac{d^2y}{dt^2} + B\frac{dy}{dt} + \omega_0^2 y = 0$$

damped.

Suppose you push down on the front end of some old motorcycle and observe that it oscillates with a period T and has an amplitude that that falls from A to fA in N complete cycles.

- 2a) (5 points) Given the observed data, what is the value of B?

$$\boxed{B = 2\sqrt{\omega_0^2 - \left(\frac{2\pi}{T}\right)^2}}$$

solution $\frac{B}{2} = C$

$y'' + 2Cy' + \omega^2 y = 0$

$\frac{-2C \pm \sqrt{4C^2 - 4\omega_0^2}}{2} < 0$ $\quad -C \pm \sqrt{(\omega_0^2 - C^2)}\, i$

ω

$\omega = \sqrt{\omega_0^2 - C^2}$

$Ae^{-\frac{B}{2}t}$

$Ae^{-\frac{B}{2}(NT)} = fA$

$\frac{2\pi}{T} = \omega$

$-\frac{B}{2}(NT) = \ln\left(\frac{fA}{A}\right)$

$$B = \frac{-2}{NT}\ln\left(\frac{fA}{A}\right)$$

$\frac{2\pi}{T} = \sqrt{\omega_0^2 - C^2}$ $\quad \left(\frac{2\pi}{T}\right)^2 = \omega_0^2 - \frac{B^2}{4}$

5

- 2b) (10 points) What is the natural (angular) frequency for the system (ω_0) in terms of the observed data?

$$\omega_0^2 = \left(\frac{2\pi}{T}\right)^2 + \frac{B^2}{4}$$

$$\omega_0 = \sqrt{\left(\frac{2\pi}{T}\right)^2 + \frac{1}{(NT)^2}\left(\ln\left(\frac{fA}{A}\right)\right)^2}$$

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- 2c) (10 points) Recall that the amplitude of a driven mass-spring system is given by:

$$A(\Omega) = \frac{\frac{F_0}{m}}{\sqrt{(\Omega^2 - \omega_0^2)^2 + (\frac{b\Omega}{m})^2}}$$

Near what angular frequency would the cycle's front end have to be driven in order to make it oscillate with large-amplitude vibrations?

resonante Ω_{res}.

$x'' + \frac{b}{m}x' + \omega_0^2 x = 0$

max: $(\Omega^2 - \omega_0^2)^2 + \left(\frac{b\Omega}{m}\right)^2 = 0$

$2(\Omega^2 - \omega_0^2)\cdot 2\Omega + 2\left(\frac{b}{m}\right)^2 \Omega = 0$

$\frac{b}{m} = B$

$\Omega\left(2(\Omega^2 - \omega_0^2) + \left(\frac{b}{m}\right)^2\right) = 0$

can't be at 0.

$\Omega^2 - \omega_0^2 = \frac{-\left(\frac{b}{m}\right)^2}{2}$

$\Omega^2 = \omega_0^2 - \frac{1}{2}\left(\frac{b}{m}\right)^2$

$$\Omega_{res} = \sqrt{\left(\frac{2\pi}{T}\right)^2 + \frac{1}{(NT)^2}\left(\ln\left(\frac{F_A}{A}\right)\right)^2 - \frac{2}{(NT)^2}\left(\ln\frac{F_A}{A}\right)^2}$$

$$= \sqrt{\left(\frac{2\pi}{T}\right)^2 - \frac{1}{(NT)^2}\left(\ln\left(\frac{F_A}{A}\right)\right)^2}$$

✓ 10

- 2d) (5 points) Now, keeping in mind that an unbalanced tire may exert a sinusoidal driving force on the suspension, what speed should you avoid if the front tire on the motorcycle has a radius R?

$\Omega = \frac{v}{\lambda}$

$v = \lambda\Omega$

X 0

3) In the picture shown above, there is a free-rolling cart, on which a barrel, of height H, diameter D, open on one end, has been mounted. There is also a one-stringed harp fashioned from a wire of mass M and length L, tied-off at both ends, hanging low from a nearby support.

The harp is plucked so that it resonates at it's N^{th} harmonic and the cart is given a good push along a path that will take it directly under the harp. It is noted that as the cart approaches the harp, the barrel resonates in a mode that has P intermediate nodes. As it recedes, the barrel resonates in an mode that has Q intermediate nodes.

- 3a) (5 points) Which is larger, P or Q? Why?

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Q P

Harp: fixed ends

N-1 int. nodes

$L = \frac{\lambda}{2}$ $L = N\frac{\lambda}{2}$

$L = 2\frac{\lambda}{2}$ $f\lambda = v$

Barrel: mixed

$f = (2N+1)\frac{v}{4H}$ No fundamental.

$L = \frac{Nv}{2f}$

$f = N \cdot \frac{v}{2L}$

Doppler effect

- 3b) (15 points) How fast is the cart moving?

$v = f\lambda$

$\left(\frac{2P+1}{2Q+1}\right) v_{sound} - v_{sound}$

$= v_{obs} + \frac{2P+1}{2Q+1} v_{obs}$

$= v_{obs}\left(1 + \frac{2P+1}{2Q+1}\right)$ $P > Q$

$\frac{f_{obs}}{f_{source}} = \frac{v_{sound} + v_{obs}}{v_{sound} + \cancel{v_{source}}}$

obs → source (+)

(1) $\dfrac{(2P+1)\frac{v_{sound}}{4H}}{f_{source}} = \dfrac{v_{sound} + v_{obs}}{v_{sound}}$

(2) $\dfrac{(2Q+1)\frac{v_{sound}}{4H}}{f_{source}} = \dfrac{v_{sound} - v_{obs}}{v_{sound}}$

P greater frequency than Q. because of this,

mixed BC:

int. nodes = N

$\frac{f_P}{f_{source}} = \frac{v_{sound} + v_{obs}}{v_{sound}}$

$\frac{f_Q}{f_{source}} = \frac{v_{sound} - v_{obs}}{v_{sound}}$

$\frac{(1)}{(2)}: \left(\frac{2P+1}{2Q+1}\right) = \frac{v_{sound} + v_{obs}}{v_{sound} - v_{obs}}$

5

n=0

n=1

n=2 $f = (2N+1)\frac{v}{4H}$

$\left(\frac{2P+1}{2Q+1}\right)(v_{sound} - v_{obs}) = v_{sound} + v_{obs}$

$$\boxed{v_{obs} = \frac{\left(\left(\frac{2P+1}{2Q+1}\right) - 1\right) v_{sound}}{\left(1 + \left(\frac{2P+1}{2Q+1}\right)\right)}}$$

$\boxed{\text{N is proportional to f for constant v so P>Q}}$

- 3c) (10 points) What is the tension in the string?

$$v = \sqrt{\frac{T}{\mu}} \qquad \mu = \frac{M}{L}$$

$$v = \sqrt{\frac{TL}{M}}$$

$$T = \frac{v^2 M}{L}$$

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$$f_{source} = N \frac{v}{2L}$$ nth harmonic

f_{source}

$$v = f_{source} \frac{(2L)}{N}$$

$$T = f_{source}^2 \frac{(4L^2)}{N^2} \frac{M}{\cancel{L}}$$

$$= f_{source}^2$$