1) When an object is immersed (fully or partly) in a fluid (or gas), it experiences an upward 'buoyant' force equal to the weight of the fluid (gas) displaced by the object.

For the following questions, when relevant, use the equation $\rho_{air}(y) = \rho_0 e^{-y/H}$, to model the volume mass density of the Earth's atmosphere, where $\rho_0$ and $H$ are characteristic constants and $y$ is the height above the Earth's surface.

- 1a) (5 points) Suppose a small hot-air balloon (of volume mass density $\rho$) is released into the atmosphere. Write the (exact) differential equation that describes the motion of the balloon.

$\Sigma F_y = ma_y$


$\rho_{air} V g - \rho V g = \rho V a_y$

$\frac{d^2y}{dt^2} = g\left(\frac{\rho_{air}}{\rho} - 1\right)$

$$\boxed{\frac{d^2y}{dt^2} + g\left(1 - \frac{\rho_0}{\rho} e^{-y/H}\right) = 0}$$

- 1b) (5 points) At what height will the balloon be in equilibrium?

In equilibrium, $\frac{d^2y}{dt^2} = 0$

$1 - \frac{\rho_0}{\rho} e^{-y/H} = 0$

$\ln(\rho/\rho_0) = -y/H$

$$\boxed{y_{eq} = H \ln\left(\frac{\rho_0}{\rho}\right)}$$

- 1c) (10 points) Define $\delta y = y - y_{eq}$, where $y_{eq}$ is the relation you found in part $b$. Execute a change of variables in which you re-write the differential equation from part $b$ in terms of $\delta y$. (Hint: the second derivative of a constant is zero).

$y = y_{eq} + \delta y \quad \leftarrow \quad y_{eq} = H\ln\left(\frac{\rho_0}{\rho}\right)$

$\frac{d^2y}{dt^2} + g\left[1 - \frac{\rho_0}{\rho} e^{-\frac{y_{eq} + \delta y}{H}}\right] = 0$

$\frac{d^2y}{dt^2} + g\left[1 - \frac{\rho_0}{\rho} e^{\ln(\rho/\rho_0)} e^{-\frac{\delta y}{H}}\right] = 0$

$$\boxed{\frac{d^2y}{dt^2} + g\left(1 - e^{-\frac{\delta y}{H}}\right) = 0}$$

- 1d) (10 points) Suppose $\delta y$ is rather small compared to $H$. Show that the balloon will oscillate around equilibrium and find the angular frequency of that oscillation.

if $\frac{\delta y}{H} \ll 1 \qquad e^{-\delta y/H} \simeq 1 - \frac{\delta y}{H}$

$\frac{d^2y}{dt^2} + g\left(\frac{\delta y}{H}\right) \simeq 0$

$\frac{d^2y}{dt^2} + \underbrace{\frac{g}{H}}_{\omega_0^2} \delta y = 0$ Simple harmonic oscillation!

$$\boxed{\omega_0 = \sqrt{\frac{g}{H}}}$$

2) The spring and shock-absorber combination attached to the front wheel of a motorcyle constitutes a damped oscillating system. If you were to push down on the handlebars of a cycle and release them, you might observe vertical oscillations that could be described by the differential equation:

$$\frac{d^2y}{dt^2} + B\frac{dy}{dt} + \omega_0^2 y = 0$$

Suppose you push down on the front end of some old motorcycle and observe that it oscillates with a period $T$ and has an amplitude that that falls from $A$ to $fA$ in $N$ complete cycles.

$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$

" $\frac{b}{2m}$ " $= \frac{B}{2}$

- 2a) (5 points) Given the observed data, what is the value of $B$?

$A = A_0 e^{-\frac{bt}{2m}}$

$fA_0 = A_0 e^{-\frac{B}{2}NT}$

$\ln(f) = -\frac{B}{2}NT$

$$\boxed{B = \frac{-2\ln(f)}{NT}}$$

- 2b) (10 points) What is the natural (angular) frequency for the system ($\omega_0$) in terms of the observed data?

$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

$\omega_0^2 = \omega^2 + \left(\frac{B}{2}\right)^2 \quad \leftarrow \quad \omega = \frac{2\pi}{T} \qquad \frac{B}{2} = -\frac{\ln(f)}{NT}$

$$\boxed{\omega_0 = \sqrt{\left(\frac{2\pi}{T}\right)^2 + \left(\frac{\ln(f)}{NT}\right)^2}}$$

- 2c) (10 points) Recall that the amplitude of a driven mass-spring system is given by:

$$A(\Omega) = \frac{\frac{F_0}{m}}{\sqrt{(\Omega^2 - \omega_0^2)^2 + (\frac{b\Omega}{m})^2}}$$

Near what angular frequency would the cycle's front end have to be driven in order to make it oscillate with large-amplitude vibrations?

$\Omega_{res} = \sqrt{\omega_0^2 - 2\left(\frac{b}{2m}\right)^2}$

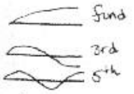
$\Omega_{res} = \sqrt{\left(\frac{2\pi}{T}\right)^2 + \left(\frac{\ln(f)}{NT}\right)^2 - 2\left(\frac{\ln(f)}{NT}\right)^2}$

$$\boxed{\Omega_{res} = \sqrt{\left(\frac{2\pi}{T}\right)^2 - \left(\frac{\ln(f)}{NT}\right)^2}}$$

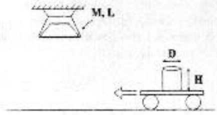
- 2d) (5 points) Now, keeping in mind that an unbalanced tire may exert a sinusoidal driving force on the suspension, what speed should you avoid if the front tire on the motorcycle has a radius $R$?

$\partial \longrightarrow V_{cm} = \Omega_{res} R$

$$\boxed{V = R\sqrt{\left(\frac{2\pi}{T}\right)^2 - \left(\frac{\ln(f)}{NT}\right)^2}}$$




for m intermediate nodes- 2m+1 'th harmonic in barrel




3) In the picture shown above, there is a free-rolling cart, on which a barrel, of height $H$, diameter $D$, open on one end, has been mounted. There is also a one-stringed harp fashioned from a wire of mass $M$ and length $L$, tied-off at both ends, hanging low from a nearby support.

The harp is plucked so that it resonates at it's $N^{th}$ harmonic and the cart is given a good push along a path that will take it directly under the harp. It is noted that as the cart approaches the harp, the barrel resonates in a mode that has $P$ intermediate nodes. As it recedes, the barrel resonates in an mode that has $Q$ intermediate nodes.

- 3a) (5 points) Which is larger, $P$ or $Q$? Why?

Higher frequencies mean shorter wavelengths mean more intermediate nodes. $f_{obs}$ is greater for approach than recession, so...

$$\boxed{P > Q}$$

- 3b) (15 points) How fast is the cart moving?

$$\frac{f_{obs}}{f_{emit}} = \frac{V_{snd} - V_{obs}}{V_{snd} - V_{src}}$$

approach $\quad \dfrac{(2P+1)\, f_{0,B}}{N\, f_{1,H}} = \dfrac{V_{snd} + |V_x|}{V_{snd}}$

recession $\quad \dfrac{(2Q+1)\, f_{0,B}}{N\, f_{1,H}} = \dfrac{V_{snd} - |V_x|}{V_{snd}}$

$$\Rightarrow \frac{2P+1}{2Q+1} = \frac{V_{snd} + |V_x|}{V_{snd} - |V_x|}$$

$$\boxed{|V_x| = V_{snd}\, \frac{P-Q}{P+Q+1}}$$

$f_{0,B} \equiv$ fund in barrel
$f_{1,H} \equiv$ fund in harp
$V_{src} = 0$
$V_{obs} = \pm |V_x|$

- 3c) (10 points) What is the tension in the string?

Use part b...

$$(2P+1)\, \frac{f_{0,B}}{N\, f_{1,H}} = 1 + \frac{|V_x|}{V_{snd}}$$

$$+ \quad (2Q+1)\, \frac{f_{0,B}}{N\, f_{1,H}} = 1 - \frac{|V_x|}{V_{snd}}$$

$$2(P+Q+1)\, \frac{f_{0,B}}{N\, f_{1,H}} = 2$$

$$f_{1,H} = \frac{(P+Q+1)}{N}\, f_{0,B}$$

$$\begin{cases} f_{0,B} = \dfrac{V_{snd}}{4H} & \text{(mixed)} \\ f_{1,H} = \dfrac{1}{2L}\sqrt{\dfrac{TL}{m}} = \dfrac{1}{2}\sqrt{\dfrac{T}{mL}} & \text{(like)} \end{cases}$$

$$\frac{1}{2}\sqrt{\frac{T}{mL}} = \frac{P+Q+1}{N}\, \frac{V_{snd}}{4H}$$

$$\boxed{T = mL \left[ \frac{P+Q+1}{N}\, \frac{V_{snd}}{2H} \right]^2}$$