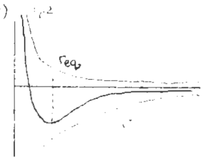
[illegible] orbits a fixed point [illegible]

[illegible] (r) is what you'd normally think of as the potential energy (that is, the energy of configu[illegible]) [illegible] system operating under an effective potential given by:

[illegible]

- 1a) (5 points) Write out the effective potential energy for the electron [illegible] nucleus. Do a quick, qualitative sketch of the effective potential [illegible] between the nucleus and the electron, and interpret the [illegible] electron's [illegible] motion.

$$\boxed{U_{eff} = \frac{L^2}{2mr^2} - \frac{Ze^2}{4\pi\epsilon_0 r}}$$



The electron is (effectively) in a stable equilibrium when it orbits at $r = r_{eq}$.

- 1b) (10 points) Using the effective potential, find the effective force on the electron [illegible] 1-dimensional case. Find the effective equilibrium value of the radial motion.

$$\frac{\partial U_{eff}}{\partial r} = \frac{-L^2}{mr^3} + \frac{Ze^2}{4\pi\epsilon_0 r^2}$$

$$F_{eff,r} = -\frac{\partial U_{eff}}{\partial r} \Rightarrow \boxed{F_{eff,r} = \frac{Ze^2}{4\pi\epsilon_0 r^3}\left[\frac{4\pi\epsilon_0 L^2}{Zme^2} - r\right]}$$

At r_{eq}, $F_{eq,r} = 0$ $\Rightarrow$ $\boxed{r_{eq} = \frac{4\pi\epsilon_0 L^2}{Zme^2}}$

$$r_{eq} = \frac{4\pi\epsilon_0 L^2}{Zme^2}$$

- 1c) (10 points) [illegible] change of variables. Let $r = r_{eq} + \delta r$. Find the effective [illegible] δr [illegible] interpret the result [illegible]

$$F_{eff,r} = \frac{Ze^2}{4\pi\epsilon_0 r^3}\left[r_{eq} - r\right]$$

$$F_{eff,r} = \frac{Ze^2}{4\pi\epsilon_0 r_{eq}^2}\left[\left(\frac{r}{r_{eq}}\right)^{-3} - \left(\frac{r}{r_{eq}}\right)^{-2}\right]$$

$$F_{eff,r} = \frac{Ze^2}{4\pi\epsilon_0 r_{eq}^2}\left[\left(1 + \frac{\delta r}{r_{eq}}\right)^{-3} - \left(1 + \frac{\delta r}{r_{eq}}\right)^{-2}\right]$$

$\frac{\delta r}{r_{eq}} \ll 1 \Rightarrow$ Taylor expand...

$$F_{eff,r} = \frac{-Ze^2}{4\pi\epsilon_0 r_{eq}^3}\,\delta r$$

$$\boxed{F_{eff,\delta r} = \frac{-Ze^2}{4\pi\epsilon_0}\left(\frac{Zme^2}{4\pi\epsilon_0 L^2}\right)^3 \delta r}$$

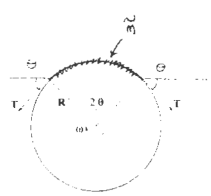
linear restoring force in δr!

The electron is sitting at the bottom of a potential well. Displace it slightly, and it will oscillate back and forth across r_{eq}...

- 1d) (5 points) Suppose the electron is displaced slightly from its nominal [illegible] Describe its subsequent motion in as much detail as [illegible]

if $F_x = -kx \Rightarrow \omega = \sqrt{k/m} \Rightarrow \boxed{\omega = \sqrt{\frac{Ze^2}{4\pi\epsilon_0 m}\left(\frac{Zme^2}{4\pi\epsilon_0 L^2}\right)^3}}$

For slight displacements from r_{eq}, the electron will oscillate back and forth about its equilibrium radius with an angular frequency equal to that boxed above...



$$\tilde{M} = M\frac{2\theta}{2\pi} = \frac{M\theta}{\pi}$$

2) A circular string of mass [illegible] uniformly distributed, and radius R rotates in a plane about a fixed point in free space (no gravity) with an angular velocity of magnitude ω.

- 2a) (10 points) Find the tension in the string by considering the net force acting on a segment that subtends an angle 2θ in the limit $\theta \to 0$.

$$\Sigma F_r = ma_r$$

$$2T\sin\theta = \tilde{m}R\omega^2$$

$$2T\sin\theta = \frac{M\theta R\omega^2}{\pi}$$

$$T = \frac{mR\omega^2}{2\pi\,\frac{\sin\theta}{\theta}} \Rightarrow \boxed{T = \frac{mR\omega^2}{2\pi}}$$

$T \propto \omega^2 \rightarrow$ seems reasonable to me

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \to 1$

- 2b) (5 points) Find the speed at which a small disturbance would propagate through the [illegible] How does this speed relate to the speed at which the string is moving in the lab? [illegible]

Pulse in a string $\quad V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{TL}{M}} = \sqrt{\frac{T\,2\pi R}{m}} = \sqrt{R^2\omega^2}$

$$\boxed{V = R\omega}$$

This is equal to the tangential speed of the rope - how cool is that?!?

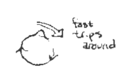
In fact, there are only three dimensionful quantities to set scale in the whole problem: M, R, ω. The only combination that can lead to the dimensions of $\frac{\text{distance}}{\text{time}}$ is $R\omega$ - so it's not too surprising...

- 2c) (5 points) When we find the speed with which disturbances propagate in a medium, with respect to what reference is that speed measured? Suppose a pulse is set in motion in our string - describe its motion as seen in the lab (there may be more than one answer).

That speed is the speed of the pulse... with respect to the medium!

- if the pulse is moving opposite the motion of the rope in the lab, the pulse will appear to be stationary in the lab (with the rope sort of pulling through it)

- if the pulse is moving with the motion of the rope, it will appear to whizz around the circumference with twice the speed (in the lab) that the rope is moving



- 2d) (5 points) Suppose we were to direct sound towards the rotating string. What frequen[cies] [illegible] hold the string [illegible]

Boundary conditions - the wave in the rope must return to each point with identical phase...

$\lambda = N\,2\pi R$ ← an integer number of wavelengths in the circumference...

$$\frac{V}{f_N} = N\,2\pi R$$

$$\frac{R\omega}{f_N} = N\,2\pi R$$

$$\boxed{f_N = N\frac{\omega}{2\pi}}$$

Again, that's kind of cool!

- But consider - given the parameters we've got, how many ways are there to construct 1/time?

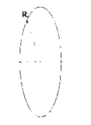


Fig 1

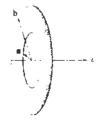


Fig 2

- [illegible] (5 points) A uniform ring of charge Q and radius R lies in the x-y plane, centered on the origin (Fig. 1). Write or derive the electric field at all points on the z-axis.

You can probably write this from memory

$$\boxed{\vec{E} = \frac{Qz}{4\pi\epsilon_0 (R^2+z^2)^{3/2}}\,\hat{k}}$$

- [illegible] (10 points) A uniform washer of charge Q lies in the x-y plane, centered on the origin, extending from $r = a$ to $r = b$ (Fig. 2). Find the electric field at all points on the z-axis.

Build the washer from rings:

$$d\vec{E} = \frac{dq\, z}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}\,\hat{k} \quad \leftarrow dq = \sigma\, 2\pi r\, dr$$

$$d\vec{E} = \frac{\sigma z\, 2r\,dr}{4\epsilon_0 (r^2+z^2)^{3/2}}\,\hat{k} \quad \leftarrow \quad u = r^2+z^2,\ du = 2r\,dr$$

$$\vec{E} = \frac{\sigma z}{4\epsilon_0}\int_{a^2+z^2}^{b^2+z^2} u^{-3/2}\,du\,\hat{k}$$

$$\vec{E} = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]\hat{k} \quad \leftarrow \sigma = \frac{Q}{\pi(b^2-a^2)}$$

$$\boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0 (b^2-a^2)}\left[\frac{1}{\sqrt{1+a^2/z^2}} - \frac{1}{\sqrt{1+b^2/z^2}}\right]\hat{k}}$$

$dq = \lambda(\phi)ds = \lambda_0 R \cos(\phi)\, d\phi$

3. A [illegible] ring of radius R and linear electric charge density $\lambda(\phi) = \lambda_0\cos(\phi)$ lies in the x-y plane, centered on the origin and spans the range $[0, \pi]$ in the usual azimuthal parameterization (shown). The 3-dimensional unit-vector shown can be written $\hat{r} = -\sin\theta\cos\phi\,\hat{i} - \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$, but [illegible] θ is not given - it will have to be re-written in terms of quantities you know.

- 3(a) (5 points) Find the electric potential (with respect to a point infinitely distant from the origin) at all points along the z-axis. How much work would you have to do to drag a test charge q from that infinitely distant point to the origin? Does it matter where that infinitely distant point is? Show your work, and explain everything as though the reader is [illegible].

$$dV = \frac{dq}{4\pi\epsilon_0 r}$$

$$dV = \frac{\lambda_0 R}{4\pi\epsilon_0}\frac{\cos\phi\, d\phi}{\sqrt{R^2+z^2}}$$

$$V = \frac{\lambda_0 R}{4\pi\epsilon_0\sqrt{R^2+z^2}}\int_0^{\pi} d\phi\cos\phi \quad \Rightarrow \quad \boxed{V = 0}$$

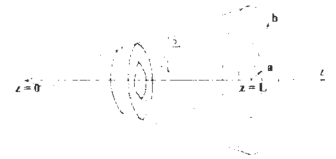
$$W_{ext} = q\Delta V = q\left[V(z) - V(\infty)\right]$$
$$= q(0-0)$$
$$= 0 \quad \Rightarrow \quad \boxed{W_{ext} = 0}$$

Because the distribution is finite, it looks the same out at all points infinitely distant ~ so where you start doesn't really matter, so long as it's very far away. Path-independence takes care of the rest...

- 3(b) (5 points) Which (rectangular) components of the electric field may be obtained from the potential you found in part a? Explain.

In part a, we calculated $V(0,0,z)$ → That is, because we plugged-in explicit values for x and y, we lost the rate-of-change information in x and y - taking the gradient of $V(0,0,z)$ won't help us find E_x & E_y...

On the other hand... we do have rate-of-change information in z, so we can find E_z ... $\left(E_z = -\frac{\partial}{\partial z}V(0,0,z)\right)$



- 2(e) (15 points) A cone of volume charge density $\rho(z) = \rho_0\frac{z^2}{L^2}$ extends symmetrically about the z-axis from its tip (at the origin) to its base (radius b, parallel to the x-y plane, a distance L from the origin). All charge is [illegible] removed from a coaxial cone of base radius $a < b$ that extends from the tip of the original cone to its base. Find the electric field (magnitude and direction) at the origin.

$$d\vec{E} = \frac{dq}{2\pi\epsilon_0 (r_2^2 - r_1^2)}\left[\frac{1}{\sqrt{1+r_1^2/z^2}} - \frac{1}{\sqrt{1+r_2^2/z^2}}\right]\hat{k} \quad \leftarrow \begin{cases} dq = \rho\, dV = \rho(z)\pi(r_2^2 - r_1^2)\,dz \\ r_1/z = a/L \\ r_2/z = b/L \end{cases} \text{similar triangles}$$

$$d\vec{E} = \frac{\rho(z)\,dz}{2\epsilon_0}\left[\frac{1}{\sqrt{1+a^2/L^2}} - \frac{1}{\sqrt{1+b^2/L^2}}\right]\hat{k} \quad \leftarrow \left\{\rho(z) = \rho_0\frac{z^2}{L^2}\right.$$

$$\vec{E} = \frac{\rho_0}{2\epsilon_0 L^2}\left[\frac{1}{\sqrt{1+a^2/L^2}} - \frac{1}{\sqrt{1+b^2/L^2}}\right]\hat{k}\int_0^L z^2\,dz$$

$$\boxed{\vec{E} = \frac{\rho_0 L}{6\epsilon_0}\left[\frac{1}{\sqrt{1+a^2/L^2}} - \frac{1}{\sqrt{1+b^2/L^2}}\right]\hat{k}}$$

- 3(c) (5 points) Is symmetry a viable option for finding the components of the electric field that you can't find using the electric potential? Explain.

Sort of. The density of negative charge mirrors the density of positive charge when reflected around the y-axis. Ultimately, this will mean $E_y = 0$. It will also probably mean that $E_x \neq 0$, but we'll have to do some work to confirm it.

- 3(d) (10 points) Find the electric field vector at all points on the z-axis.

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$$

$$d\vec{E} = \frac{\lambda_0 R\cos\phi\, d\phi}{4\pi\epsilon_0 (R^2+z^2)}\left(-\sin\theta\cos\phi\,\hat{i} - \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right)$$

$$\vec{E} = \frac{\lambda_0 R}{4\pi\epsilon_0 (R^2+z^2)}\int_0^{\pi} d\phi\left(-\sin\theta\cos^2\phi\,\hat{i} - \sin\theta\cos\phi\sin\phi\,\hat{j} + \cos\theta\cos\phi\,\hat{k}\right)$$

$$\vec{E} = \frac{-\lambda_0 R}{4\pi\epsilon_0 (R^2+z^2)}\sin\theta\,\frac{\pi}{2}\,\hat{i} \quad \leftarrow \sin\theta = \frac{R}{\sqrt{R^2+z^2}}$$

$$\vec{E} = \frac{-\lambda_0\pi R^2}{8\pi\epsilon_0 (R^2+z^2)^{3/2}}\,\hat{i}$$

$$\boxed{\vec{E} = \frac{-\lambda_0 R^2}{8\epsilon_0 (R^2+z^2)^{3/2}}\,\hat{i}}$$

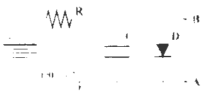
- 3(e) (5 points) Describe the electric field as it would appear at points quite distant from the origin. Explain the behavior of the field, with regard to distance from the [illegible], as [illegible] as you can (hint: compare the dependence on distance to what you would see with a point charge).

For points far from the origin, $z \gg R$ and $|\vec{E}| \propto \frac{1}{z^3}$

The field from a point-charge should fall off as $1/z^2$,

So what we're seeing here is falling off faster --

At a great distance, the field looks like that of a dipole!

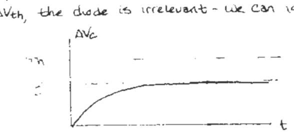


[illegible] the circuit shown above [illegible] has been open a long time, and the [illegible] component [illegible] forward-biased [illegible] in the direction of the arrow. It has no resistance when the potential difference across it exceeds the threshold voltage for the device (ΔV_{th}). When the potential difference across the device is less than the threshold voltage, it has infinite resistance.

- 5a) (5 points) How will the maximum potential difference across the capacitor compare to the potential difference across the battery? Suppose $\mathcal{E} < \Delta V_{th}$. Find ΔV_{AB} as a function of time and plot it qualitatively. Sketch $\mathcal{E}$ and ΔV_{th} on your plot.

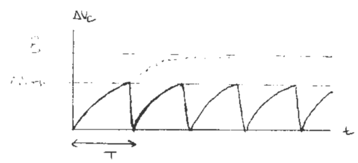
$\Delta V_{c,max} \leq \mathcal{E}$, So... if $\mathcal{E} < \Delta V_{th}$, the diode is irrelevant - we can ignore it...

$$\boxed{\Delta V_{AB} = \Delta V_C = \mathcal{E}(1-e^{-t/RC})}$$



- 5b) (5 points) Suppose $\mathcal{E} > \Delta V_{th}$. Describe what happens now. Plot ΔV_{AB} qualitatively as a function of time. Sketch $\mathcal{E}$ and ΔV_{th} on your plot.

if $\mathcal{E} > \Delta V_{Th}$, the capacitor will discharge every time ΔV_C reaches ΔV_{Th}...



- 5c) (5 points) What is the period of the waveform you sketched [illegible]?

$$\Delta V_{Th} = \mathcal{E}(1-e^{-t/RC})$$

$$1-\frac{\Delta V_{Th}}{\mathcal{E}} = e^{-t/RC}$$

$$\Rightarrow \boxed{T = -RC\ln\left(\frac{\mathcal{E}-\Delta V_{Th}}{\mathcal{E}}\right)}$$

- 5d) (5 points) Suppose you wanted the potential difference on the [illegible] A and B to have a short period and a very linear rise. How would you choose [illegible]? Explain.

if $t/RC \ll 1$, $e^{-t/RC} \simeq 1-\frac{t}{RC}$ and $\Delta V_{AB} \simeq \frac{\mathcal{E}}{RC}t$ (linear!)

But large RC means long period unless $\Delta V_{Th} \simeq \mathcal{E}$

$\Rightarrow$ we want a large value of RC and $\mathcal{E}$ slightly larger than ΔV_{Th}

- 5e) (10 points) In real life some sort of load resistance will have to be connected across terminals A and B. How will that affect the period (for full credit, write the new expression for the period, and explain)? Are there any values of load resistance that [illegible] particularly problematic? Explain.

We've Solved this Circuit before, without the diode...

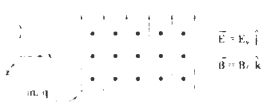
$\Delta V_{c,max}$ will go from $\mathcal{E}$ to $\mathcal{E}\frac{R_L}{R+R_L}$

and RC will go to $\frac{RR_L}{R+R_L}C$...

$$T \to -\frac{RR_LC}{R+R_L}\ln\left(\frac{\mathcal{E}R_L - \Delta V_{Th}(R+R_L)}{\mathcal{E}R_L}\right)$$

if $\mathcal{E}R_L < \Delta V_{Th}(R+R_L)$, we've got problems (the capacitor never discharges)∴ we'll need to make sure

$$R_L > \frac{\Delta V_{Th}R}{\mathcal{E}-\Delta V_{Th}}$$



6) A particle of mass m and charge q is released at rest at the origin in a region filled with uniform electric and magnetic fields as shown.

- 6a) (5 points) Find the Lorentz force on the particle at an instant when its velocity is given [illegible]

$$\vec{F_L} = q(\vec{E}+\vec{v}\times\vec{B})$$

$$\vec{F_L} = q(E_y\hat{j}+B_zv_y\hat{i}-B_zv_x\hat{j})$$

$$\boxed{\vec{F_L} = qB_zv_y\hat{i}+q(E_y-B_zv_x)\hat{j}}$$

- 6b) (5 points) Use Newton's second law to find (coupled) equations of motion for the particle (in differential form). Do not decouple the equations [illegible] the next part.

$$\sum\vec{F} = m\vec{a}$$

$$qB_zv_y = m\frac{dv_x}{dt}$$

$$q(E_y-B_zv_x) = m\frac{dv_y}{dt}$$

$$0 = m\frac{dv_z}{dt}$$

$$\Rightarrow \boxed{\begin{aligned}\frac{dv_x}{dt} - \frac{qB_z}{m}v_y &= 0\\ \frac{dv_y}{dt} + \frac{qB_z}{m}v_x &= \frac{qE_y}{m}\\ \frac{dv_z}{dt} &= 0\end{aligned}}$$

(6c)

* To decouple, we'll need derivatives of these equations - let's grab them while we're here...

$$\frac{d^2v_x}{dt^2} - \frac{qB_z}{m}\frac{dv_y}{dt} = 0$$

$$\frac{d^2v_y}{dt^2} + \frac{qB_z}{m}\frac{dv_x}{dt} = 0$$

$$\Rightarrow \frac{dv_x}{dt} = \frac{-m}{qB_z}\frac{d^2v_y}{dt^2}, \quad \frac{dv_y}{dt} = \frac{m}{qB_z}\frac{d^2v_x}{dt^2}$$

→ use these in part C

- 6c) (10 points) Now...decouple the differential equations. By the end of this part you should have separate equations in v_x, v_y and v_z.

use the derivatives on the preceeding page to decouple the equations...

$$\boxed{\begin{aligned}\frac{d^2v_x}{dt^2} + \left(\frac{qB_z}{m}\right)^2 v_x &= \frac{q^2B_zE_y}{m^2}\\ \frac{d^2v_y}{dt^2} + \left(\frac{qB_z}{m}\right)^2 v_y &= 0\\ \frac{dv_z}{dt} &= 0\end{aligned}}$$

- 6d) (10 points) Assume the particle starts at rest and find v_x, v_y and v_z as functions of t. Hints: remember how you used initial conditions to find amplitude for a simple harmonic oscillator and how you dealt with shifts in the equilibrium position.

General Solutions follow from our work with oscillators...

$v_x(t) = v_{x,max}\cos(\omega t+\phi)+\frac{E_y}{B_z}$ $\quad$ $v_y(t) = v_{y,max}\sin(\omega t+\delta)$ $\quad$ $v_z(t) =$ Constant

$a_x(t) = -v_{x,max}\,\omega\sin(\omega t+\phi)$ $\quad$ $a_y(t) = v_{y,max}\,\omega\cos(\omega t+\delta)$ $\quad$ $a_z(t) = 0$

$\omega = \frac{qB_z}{m}$

initial Conditions: $v_{ox} = 0$, $v_{oy} = 0$, $v_{oz} = 0$

$a_{ox} = \frac{F_{oL,x}}{m} = 0$, $a_{oy} = \frac{F_{oL,y}}{m} = \frac{qE_y}{m}$, $a_{oz} = \frac{F_{oL,z}}{m} = 0$

$\Rightarrow v_z(t) = 0$ (easy)

$\Rightarrow v_y(0) = 0 \Rightarrow \delta = 0$

$a_y(0) = \frac{qE_y}{m} \Rightarrow v_{y,max} = \frac{E_y}{B_z}$

$\Rightarrow a_x(0) = 0 \Rightarrow \phi = 0$

$v_x(0) = 0 \Rightarrow v_{x,max} = -\frac{E_y}{B_z}$

$$\Rightarrow \boxed{\begin{aligned}v_x(t) &= \frac{E_y}{B_z}\left(1-\cos(\omega t)\right)\\ v_y(t) &= \frac{E_y}{B_z}\sin(\omega t)\\ v_z(t) &= 0\\ \text{where } \omega &= \frac{qB_z}{m}\end{aligned}}$$