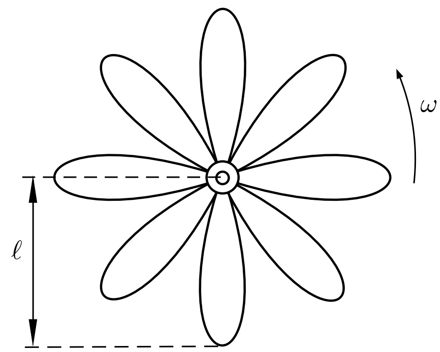
Physics 1A - Winter 2016

Lecture 2

Final Exam

Problem 1.



A submarine moves forward because it spins a propeller which displaces water backward. The propeller is spun by a crankshaft that is connected to the submarine's engine. The propeller has N blades (for example in the diagram $N = 8$) of length ℓ. Because of fluid resistance, the water exerts an angular speed-dependent force $f = c\omega^2$ on each blade, tangent to the circle in which the blade is rotating. You can treat this force as though it's acting at the center of each blade. When the propeller spins, the resulting magnitude a of the ship's acceleration is related to the angular speed ω of the propeller as $a = gN\omega/\omega_0$ where ω_0 is a given constant.

If τ is the magnitude of the maximum torque the engine can exert on the crankshaft, and if this torque gives the submarine a maximum acceleration of $g/4$, how many blades does its propeller have?

Solution. Let the positive z-direction point out of the page, so the axis of rotation of the propeller is along the z-axis. The propeller is a rigid body rotating around a fixed axis, so we can apply the rotational analog of Newton's Second Law to this scenario:

$$\tau_{\text{net},z} = I\alpha_z. \tag{1}$$

Recall from the statement of the problem that the acceleration of the submarine increases linearly with the angular velocity of the propeller. This means that if the submarine has reached maximum acceleration, $a_{\max}$, then the propeller is spinning at some maximum angular velocity $\omega_{\max}$, and these two are related as follows:

$$a_{\max} = gN\frac{\omega_{\max}}{\omega_0} \tag{2}$$

We are told that the maximum acceleration of the sub is $g/4$, so this gives

$$\frac{g}{4} = gN\frac{\omega_{\max}}{\omega_0}. \tag{3}$$

Moreover, when the submarine's propeller reaches its maximum angular velocity, the angular acceleration of the propeller will be zero, since otherwise the angular velocity would increase, which it can't. It follows that when the propeller reaches $\omega_{\max}$, the net torque $\tau_{\text{net},z}$ vanishes (equals zero). On the other hand, the net torque consists of two pieces: the torque exerted by the shaft on the propeller, and the torque exerted by the surrounding water slowing it down. The net torque vanishing means that these must sum to zero:

$$\tau_{\text{shaft},z} + \tau_{\text{water},z} = 0. \tag{4}$$

Now suppose for concreteness that the propeller is spinning counterclockwise, then since $\omega_{\max}$ has been reached, and since the torque of the shaft on the propeller is causing this counterclockwise rotation, it must point in the positive z-direction. This means

$$\tau_{\text{shaft},z} = \tau_{\max} \tag{5}$$

Moreover, the torque of the surrounding water will be the sum of all torques on each propeller, each of which has magnitude $f_{\max}(\ell/2)$ since as told in the problem statement, we can treat the force f of the water on the propeller as acting at its center. Noting also that this torque will be opposite in direction to the torque exerted by the shaft, we find

$$\tau_{\text{water},z} = -N f_{\max}\frac{\ell}{2} = -\frac{Nc\omega_{\max}^2\ell}{2}. \tag{6}$$

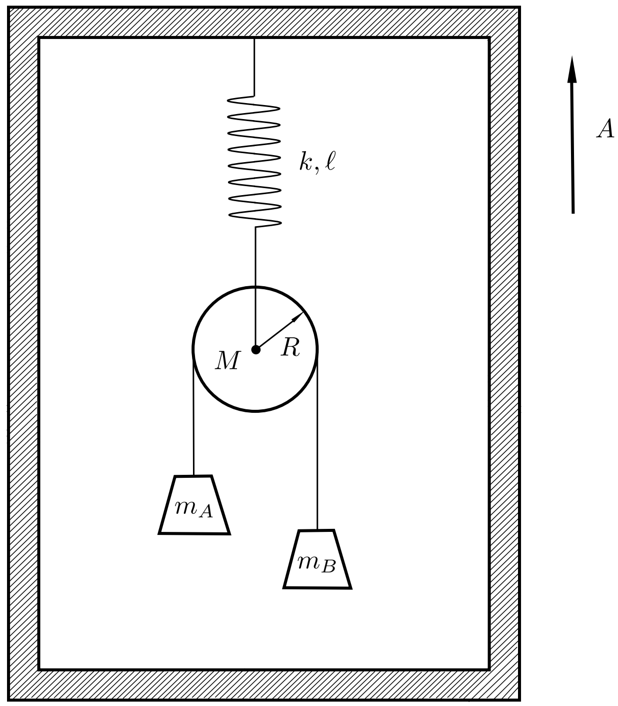
Combining these torque results gives

$$\tau_{\max} - \frac{Nc\omega^2\ell}{2} = 0. \tag{7}$$

Equations (3) and (7) form a system of two equations in the two unknowns N and $\omega_{\max}$. Solving these equations for N gives

$$\boxed{N = \frac{c\omega_0^2\ell}{32\tau_{\max}}} \tag{8}$$

Problem 2.



An Atwood's machine with a pulley in the shape of a uniform disk of mass M and radius R is in an elevator having vertical acceleration A relative to the ground. Let positive A correspond to the elevator accelerating upward. Mass m_A hangs on the left side of the pulley while mass m_B hangs on the right. The rope connecting masses m_A and m_B is massless. The pulley is suspended from the ceiling of the elevator by a spring of spring constant k and natural length ℓ.

(a) Consider the special case $m_A = m_B$. What would you expect the length of the spring to be in the following limits:

i. $A \to 0$

ii. $A \to -g$

iii. $A \to g$

Justify your answers using physical reasoning and minimal math if possible.

(b) When $m_A \neq m_B$ but $A = 0$, would you expect the tension in the spring to be less than, equal to, or greater than the weight of the pulley plus the weight of the masses hanging from the pulley? Justify your answers using physical reasoning and minimal math if possible.

(c) Determine a general expression for the length of the spring in terms of the given variables.

(d) Does your mathematical answer from part (c) agree with your answers from part (a) in each special case? If so, demonstrate this mathematically for each case. If not, you may want to re-evaluate your intuition, or your mathematical answer, or both.

(e) Does your mathematical answer from part (c) agree with your answer from part (b)? If so, demonstrate this mathematically for each case. If not, you may want to re-evaluate your intuition, or your mathematical answer, or both.

Solution.

(a) The length of the spring depends on the tension in the spring. When the tension or compression in the spring is zero, the spring is at its natural length, but when the tension its tension or compression is nonzero, the length of the spring is greater than or less than the natural length respectively.

When $m_A = m_B$, neither mass will accelerate relative to the elevator, and for the purposes of analyzing the tension or compression of the spring, we lose no generality in considering the case that they are at rest relative to the elevator. In that case, it's simply as though there is a weight $m_A + m_B + M$ hanging from the spring. For simplicity, we use the notation $m_A = m$ and $m_B = m$ in this case that the masses are equal.

i. When $A \to 0$, the force on the spring is the same as if the elevator were just at rest on the ground. Since we already remarked that when $m_A = m_B$, it's as though there is a weight $2m + M$ hanging from the spring, we would expect the length L of the spring to obey $k(L - \ell) = (2m + M)g$ and therefore the length of the spring would be

$$L \to \boxed{\ell + \frac{2m + M}{k} g} \tag{9}$$

ii. When $A \to -g$, the elevator is in free fall. In this case, the spring feels no force whatsoever, so we expect the length of the spring to be its natural length: $L \to \boxed{\ell}$.

iii. When $A \to g$, it feels on the inside the elevator as though gravity is twice as strong. Therefore, the answer is the same as in part (i) except with $g \to 2g$:

$$L \to \boxed{\ell + \frac{2m + M}{k}(2g)} \tag{10}$$

(b) For the case $m_A \neq m_B$ but $A = 0$, imagine the elevator sitting at rest at a certain floor, then we have a standard Atwood's machine with massive pulley. We can get intuition for what happens in the case $m_A \neq m_B$, consider the extreme case $m_A \to 0$

but $m_B > 0$, then this reduces to the case of m_B falling while wrapped around a massive pulley. In this case, the tension in the spring equals the weight of the pulley plus the tension in the rope holding m_B, but the tension in the rope holding m_B will be less than the weight of m_B because otherwise, m_B wouldn't fall down. Hence, the tension in the spring is <u>less than</u> the sum of the weights of the objects hanging from it.

(c) Once we find the force F in the spring, we can find its length. Drawing free body diagrams for masses m_A, m_B, and the pulley in the y-direction gives the following Newton's Second Law equations:

$$T_A - m_A g = m_A a_{A,y} \tag{11}$$
$$T_B - m_B g = m_B a_{B,y} \tag{12}$$
$$F - Mg - T_A - T_B = M a_{P,y} \tag{13}$$

This is three equations in six unknowns $T_A, T_B, F, a_{A,y}, a_{B,y}, a_{P,y}$. We need more equations. We firstly have the following constraints:

$$a_{A,y} + a_{B,y} = 2a_{P,y} \tag{14}$$
$$a_{P,y} = A. \tag{15}$$

We obtain another by applying the analog of Newton's Second Law to the rotation of the pulley about its center of mass:

$$T_A R - T_B R = \frac{1}{2} M R^2 \alpha_z \tag{16}$$

Finally we note the constraint that the angular acceleration of the pulley is related to the acceleration of mass m_B *relative to the elevator* in the y-direction as $a_{B/E,y} = R\alpha_z$, but since the acceleration of the elevator relative to the ground is A, we find that $a_{B,y} = a_{B/E,y} + A$ so that

$$a_{B,y} - A = R\alpha_z. \tag{17}$$

If we combine all equations we've discovered thus far, we can solve for F. The rest of the solution is purely algebraic manipulation – the physics is complete. First, we utilize and eliminate all of the constraints. For notational simplicity we make the definition $a_{B,y} = a$, then using all constraints gives

$$T_A - m_A g = m_A(2A - a) \tag{18}$$
$$T_B - m_B g = m_B a \tag{19}$$
$$F - Mg - T_A - T_B = MA \tag{20}$$
$$T_A - T_B = \frac{1}{2} M(a - A) \tag{21}$$

Subtracting the second equation from the first and then using the fourth equation gives the following equation that can be solved for A:

$$\frac{1}{2}M(a-A) - m_A g + m_B g = 2m_A A - m_A a - m_B a. \tag{22}$$

Solving for a gives

$$a = \frac{m_A(g+2A) - m_B g + \frac{1}{2}MA}{m_A + m_B + \frac{1}{2}M} \tag{23}$$

We can plug this back into the NSL equatins for masses m_A and m_B to obtain the tensions T_A and T_B:

$$T_A = m_A(g + 2A - a) \tag{24}$$

$$= m_A \frac{(m_A + m_B + \frac{1}{2}M)(g+2A) - m_A(g+2A) + m_B g - \frac{1}{2}MA}{m_A + m_B + \frac{1}{2}M} \tag{25}$$

$$= \frac{m_A(2m_B + \frac{1}{2}M)}{m_A + m_B + \frac{1}{2}M}(g+A). \tag{26}$$

and

$$T_B = m_B(g + a) \tag{27}$$

$$= m_B \frac{(m_A + m_B + \frac{1}{2}M)g + m_A(g+2A) - m_B g + \frac{1}{2}MA}{m_A + m_B + \frac{1}{2}M} \tag{28}$$

$$= \frac{m_B(2m_A + \frac{1}{2}M)}{m_A + m_B + \frac{1}{2}M}(g+A) \tag{29}$$

it follows that the force F in the spring is

$$F = M(g+A) + T_A + T_B \tag{30}$$

$$= \left(M + \frac{4m_A m_B + \frac{1}{2}M(m_A + m_B)}{m_A + m_B + \frac{1}{2}M}\right)(g+A) \tag{31}$$

One we know the force in the spring, we can use the fact that this force determines the length L of the spring according to

$$F = k(L - \ell) \tag{32}$$

and we find

$$L = \boxed{\ell + \frac{1}{k}\left(M + \frac{4m_A m_B + \frac{1}{2}M(m_A + m_B)}{m_A + m_B + \frac{1}{2}M}\right)(g+A)} \tag{33}$$

(d) When $m_A = m_B = m$, the expression for the length of the spring vastly simplifies:

$$L = \ell + \frac{1}{k}\left(M + \frac{4m^2 + Mm}{2m + \frac{1}{2}M}\right)(g + A) \tag{34}$$

$$= \ell + \frac{1}{k}(M + 2m)(g + A) \tag{35}$$

i. When $A \to 0$, our mathematical answer gives

$$L \to \ell + \frac{1}{k}(M + 2m)g \tag{36}$$

which agrees with our predicted answer.

ii. When $A \to -g$, we obtain $L = \ell$ which agrees with our predicted answer.

iii. When $A \to g$, we obtain

$$L \to \ell + \frac{1}{k}(M + 2m)(2g) \tag{37}$$

which agrees with our predicted answer. PHEW!

(e) In the case $m_A \neq m_B$ but $A \to 0$, we obtained intuition for what would happen in the further special case that one of the masses was zero. Let's say that $m_A \to 0$ for example, then our answer reduces to

$$L \to \ell + \frac{1}{k}\left(M + \frac{\frac{1}{2}Mm_B}{m_B + \frac{1}{2}M}\right)g \tag{38}$$

$$= \ell + \frac{1}{k}\left(M + \frac{\frac{1}{2}M}{m_B + \frac{1}{2}M}m_B\right)g \tag{39}$$

$$< \ell + \frac{M + m_B}{k}g \tag{40}$$

This agrees with our intuition that the tension in the spring will be less than if the weight $(M + m_B)g$ were hanging from the pulley.

Problem 3.

Nancy is spinning on an ice rink (effectively frictionless) at a certain angular speed ω. Josh throws her a water bottle, and she catches it. Let the word "system" refer to the bottle + Nancy. Let the z-direction point vertically, away from the ice and perpendicular to it.

For all of the following questions, consider the time interval from the moment just after Josh lets the water bottle go, to the moment just after Nancy catches it.

(a) Is the total linear momentum of the system conserved in the z-direction? Justify mathematically, and explain the math in words.

(b) Is the total linear momentum of the system conserved in the x-y-direction? Justify mathematically, and explain the math in words.

(c) Is the total angular momentum of the system conserved in the z-direction? Justify mathematically, and explain the math in words.

(d) Is the total angular momentum of the system conserved in the x-y-direction? Justify mathematically, and explain the math in words.

Solution.

(a) If the net external force in the z-direction is zero, then the total momentum of the system in the z-direction is conserved, otherwise, it's not. When the water bottle is flying to Nancy, it is acted on by gravity, and during that time there is a net z-component of force due to gravity on the water bottle, so the momentum of the system in the z-direction is <u>not conserved</u>.

(b) After the water bottle is released, there are no external forces on the system in the x-y-direction, so the total momentum of the system in this plane <u>is conserved</u>.

(c) If the net external torque in the z-direction is zero, then the total angular momentum in the z-direction is conserved. Since All force on the system are vertical (normal and gravity), the external torques on the system are perpendicular to the z-direction, so the z-component of the net external torque is zero. Thus, the z-component of the total angular momentum of the system <u>is conserved</u>.

(d) When the water bottle is in the air, there is a net external torque on the system in the x-y plane due to gravity as noted in part (a), so the angular momentum of the system in this direction is <u>not conserved</u>.

Problem 4.

A large-diameter bowl of mass M sits on a cooking scale. A small amount of water of mass m is slowly poured into the bowl out of a cup from a height h above the bottom of the bowl. It takes a time T from the moment when the water first hits the bottom of the bowl to the moment when all of the water has flowed in. The water flows in at a constant mass per unit time. The bowl starts empty. Let $t = 0$ be the moment at which the water first strikes the bowl.

(a) As t approaches T from values less than T, will the scale show a weight less than, equal to, or greater than $(M + m)g$? Use physical reasoning and minimal math if possible.

(b) Determine an expression for the weight as a function of time that the scale reads from $t = 0$ until the moment when the last of the water strikes the bowl?

(c) Does your answer in part (b) agree with your answer in part (a)? Explain.

(d) Is there a time when the scale reads a weight $(M + m)g$? If not, explain why not. If so, determine an expression for this time.

Solution.

(a) In the limit that $t \to T$ from values less than T, all of the mass of the water will have poured into the bowl, but there will still be a force of the water flowing into the bowl as well, so one would expect a weight greater than $(M + m)g$.

(b) This is a mass flow problem. We apply the mass flow equation to the bowl with the water flowing into it in the vertical direction along which the water is falling which we call the y-direction:

$$F_{\text{ext},y} = M(t)\frac{dv}{dt}(t) - \frac{dM}{dt}(t)u_y. \tag{41}$$

Since the bowl is at rest on the scale, $dv/dt = 0$, so the first term on the right vanishes. Since a mass m falls into the bowl in a time T at a constant rate, the mass flow rate $dM/dt = m/T$. Since the water falls into the bowl from a height h, kinematics tells us that it hits the bowl with a velocity $u_y = -\sqrt{2gh}$ (where we have taken the positive y-direction to point upward). The force of the scale on the bowl is a normal force N in the positive direction (this is what the scale reads). There is also a gravitational force on the bowl plus the mass in it at time t given by $(M + (m/T)t)g$. These two forces combine to give the net external force in the y-direction. Putting all of these observations together into the mass flow equation gives

$$N - \left(M + \frac{m}{T}t\right) g = -\frac{m}{T}(-\sqrt{2gh}). \tag{42}$$

It follows that the weight the scale reads as a function of time while the water is falling is

$$N = \boxed{\left(M + \frac{m}{T}t\right) g + \frac{m}{T}\sqrt{2gh}} \tag{43}$$

(c) In the limit $t \to T$, the weight the scale reads becomes

$$N \to (M+m)g + \frac{m}{t}\sqrt{2gh} > (M+m)g \tag{44}$$

As expected, the scale reads a force greater than the combined weight of the falling water and the bowl. In fact, the extra term is precisely the "thrust" force of the falling water.

(d) There is a time when the water is falling that the scale reads $(M+m)g$ which we can solve for by using our answer to part (b) and setting $F = (M+m)g$:

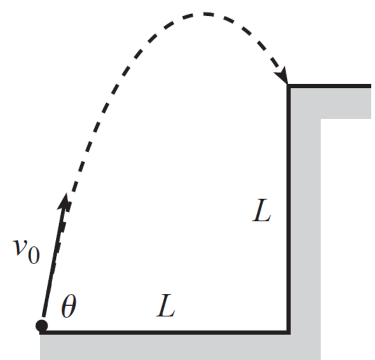
$$(M+m)g = \left(M + \frac{m}{T}t\right) g + \frac{m}{T}\sqrt{2gh} \tag{45}$$

Solving for t gives

$$t = \frac{T}{m}\left(M + m - \frac{m}{T}\sqrt{\frac{2h}{g}} - M\right) \tag{46}$$

$$= \boxed{T - \sqrt{\frac{2h}{g}}}. \tag{47}$$

Problem 5.



A small ball is launched from the ground onto the corner of a cliff as shown.

(a) For which launch angles θ in the range $[0, \pi]$ is it possible for the ball to hit the corner? Explain in the most convincing way you can.

(b) For a given angle θ that does allow the ball to hit the corner, what launch speed v_0 is necessary for the ball to precisely hit the corner?

(c) Examine the limiting cases $\theta \to \pi/2$ and $\theta \to \pi/4$. What happens to the required launch speed in each of these cases according to the formula you derived from part (b)? Does the behavior of your formula make sense in these limiting cases? Explain with physical reasoning.

Solution.

(a) We first enumerate the launch angles for which the ball cannot hit the cliff. What remains will be the launch angles that allow the ball to hit the corner. If $\theta > \pi/2$, then the ball will be launch in the direction opposite the cliff, so it certainly will not be able to hit the corner. When $\theta = \pi/2$. The ball also will not be able to hit the cliff because it will be traveling vertically upward and then back down again. When $\theta \leq \pi/4$, then the ball cannot hit the corner either. Too see why, notice that if the ball is launched with near infinite speed, then it will travel in an essentially straight line, so if it's launched very fast at an angle just less than $\pi/4$, then it can reach the corner. But for any lower angle, the ball's trajectory will always curve downward and will either hit the wall of the cliff or the ground before reaching the horizontal position of the corner. Hence, the values of θ for which the ball can hit the corner are $\boxed{\pi/2 < \theta < \pi/4}$.

(b) If we take the launch point to be the origin, then the x- and y-positions of the ball as a function of time will be

$$x = v_0 \cos\theta t \tag{48}$$

$$y = v_0 \sin\theta t - \frac{1}{2}gt^2. \tag{49}$$

If we want the ball to hit the corner, then we want its trajectory to intersect the point $x = L$ and $y = L$. Plugging these equations into the equations above, and using the x-equation to eliminate t, we obtain the following equation for v_0:

$$L = L\tan\theta - \frac{1}{2}g\left(\frac{L}{v_0\cos\theta}\right)^2 \tag{50}$$

Solving for v_0 gives

$$v_0 = \boxed{\sqrt{\frac{gL}{2\cos\theta(\sin\theta - \cos\theta)}}} \tag{51}$$

(c) When $\theta \to \pi/2$, we have $\cos\theta \to 0$ and $\sin\theta \to 1$, so $v_0 \to \infty$. This agrees not only with the intuition that at $\theta = \pi/2$, the ball cannot hit the corner, but also with the additional intuition that as the launch angle goes close to vertical, a higher speed is necessary for the very small horizontal component to allow the ball to even get to the corner. When $\theta \to \pi/4$, we have $\cos\theta \to 1/\sqrt{2}$ and $\sin\theta \to 1/\sqrt{2}$. This implies again that $v_0 \to \infty$ since there is a $\sin\theta - \cos\theta$ term in the denominator. This again agrees with the intuition that if we want to through the ball straight at the corner, then we need its trajectory not to curve downward much relative to gravity, so we need to throw it very hard.