Problem 1 (3+3+4+10)

The displacement $u(t)$ of a dynamic system is governed by the following second order, linear, differential equation (SLDE)

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad u(0) = u_0,\ \dot{u}(0) = \dot{u}_0$$

where
- m is the mass.
- c is the damping constant,
- k is the spring constant, and
- $F(t)$ is the externally applied force

(a) Define the following in terms of the displacement $u(t)$ and structural parameters m, c, k

- steady state response of the system
- critically damped system

(b) Using linearity of the solution, write down the general form of the solution in terms of $u_0,\ \dot{u}_0$ and $F(t)$

(c) If $m = 8$ lb, $k = 16\ \frac{\text{lb}}{\text{ft}},\ c = \frac{1}{4}\ \frac{\text{lb-sec}}{\text{ft}},\ F(t) = 4\cos(2t)$ lb, find the amplitude of the steady state response. Do not use any formula.

1 $m\ddot{u} + c\dot{u} + ku = F(t)$ $\quad u(0) = u_0$, $\dot{u}(0) = \dot{u}_0$

a Steady State Response

$\lim_{t \to \infty} u(t)$: steady state response $= u_p(t)$

* Critically damped system

$$m\ddot{u} + c\dot{u} + ku = F(t)$$

$$\therefore \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{F}{m}(t)$$

$$\therefore \ddot{u} + 2b\dot{u} + \omega^2 u = \frac{F}{m}(t) \quad \text{where } 2b = \frac{c}{m}, \quad \omega^2 = \frac{k}{m}$$

Homog equ.

$$\ddot{u} + 2b\dot{u} + \omega^2 u = 0$$

Let $u = e^{rt}$

AE $\quad r^2 + 2br + \omega^2 = 0$

$$\therefore r = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

for critically damped system

$$4b^2 - 4\omega^2 = 0$$

$$\frac{c^2}{m^2} - 4\frac{k}{m} = 0$$

$$\therefore c^2 = 4km.$$

& r_1 & $r_2 = -b, -b = \frac{-c}{2m}, -\frac{c}{2m}$

$$\therefore u_h(t) = C_1 e^{-(c/2m)t} + C_2 t e^{-(c/2m)t}$$

$$\therefore u_{gen}(t) = C_1 e^{-(c/2m)t} + C_2 t e^{-(c/2m)t} + u_p(t)$$

b/ $v(t) = u_0 \phi_1(t) + \dot{u}_0 \phi_2(t) + \int_0^t F(\tau)\, \phi_3(t-\tau)\, d\tau$

where ϕ_1, ϕ_2 & ϕ_3 are structural functions

1c

$m = 8$ lb

$k = 16$ lb/ft

$c = \frac{1}{4} \frac{\text{lb sec}}{\text{ft}}$

$F(t) = 4\cos(2t)$

$m\ddot{u} + c\dot{u} + ku = F(t)$

$\therefore 8\ddot{u} + \frac{1}{4}\dot{u} + 16u = 4\cos 2t$.

$\therefore$ Homog. eqn

$8\ddot{u} + \frac{1}{4}\dot{u} + 16u = 0$

or, $32\ddot{u} + \dot{u} + 64u = 0$

Let $u = e^{rt}$

$\therefore$ AE: $32r^2 + r + 64 = 0$.

$\therefore r = \frac{-1 \pm \sqrt{1 - 4.32.64}}{64} = \frac{-1 \pm \sqrt{-8191}}{64} = \frac{-1 \pm i\,90.5}{64}$

$= -0.016 \pm i\,1.41$

$\therefore u_h = c_1 e^{-0.016t}\cos(1.41t) + c_2 e^{-0.016t}\sin(1.41t)$.

$g(t) = 4\cos 2t$.

$\therefore y_p(t) = A\cos 2t + B\sin 2t$.

$\therefore y_p'(t) = -2A\sin 2t + 2B\cos 2t$

$y_p''(t) = -4A\cos 2t - 4B\sin 2t$.

Putting in the DE,

$$-32A\cos 2t - 32B\sin 2t - \frac{1}{2}A\sin 2t + \frac{1}{2}B\cos 2t$$

$$+16A\cos 2t + 16B\sin 2t = 4\cos 2t$$

Comparing coeff

$\underline{\cos 2t}$

$$-32A + \frac{1}{2}B + 16A = 4$$

$$\therefore -16A + \frac{1}{2}B = 4$$

$$\therefore -32A + B = 8$$

$$\therefore B = 8 + 32A \quad \text{— i}$$

$\underline{\sin 2t}$

$$-32B - \frac{1}{2}A + 16B = 0.$$

$$\therefore -16B - \frac{1}{2}A = 0.$$

$$\therefore -32B - A = 0.$$

$$\therefore B = -\frac{A}{32} \quad \text{— ii}$$

$$\therefore 8 + 32A = -\frac{A}{32}$$

$$\therefore 256 + 1024A = -A$$

$$\therefore 1025A = -256$$

$$\therefore A = -0.25$$

$$\therefore B = 0.0078.$$

$$\therefore u_p(t) = -0.25\cos 2t + 0.0078\sin 2t.$$

$\therefore$ Amplitude of steady state response

$$= \sqrt{(-0.25)^2 + (0.0078)^2} = 0.25$$

Problem 2 (6+6+8)

a. Explain the method of undetermined coefficients used to determine the particular solution y_p of a given nonhomogeneous SLDE.

Is the method applicable to any general SLDE?

Does the form of y_p depend on the solution of the corresponding homogenous problem?

b. **Write down the forms of the particular solution** of the differential equation $y'' - 4y' + 4y = g(t)$ using the method of undetermined coefficients for the following cases

1. $g(t) = t\sin(t)e^{t}$
2. $g(t) = \sin(t) + te^{2t}$
3. $g(t) = t + \cos(t) + e^{2t}$

For the third case above, derive the **complete general solution** of the SLDE.

Problem 2.

a.
- find homogeneous solution
- "guess" particular solution given $g(t)$
  → if there is repetition between guessed-y_p & homogeneous solutions, multiply guessed-y_p by t until there is no repetition.
- differentiate final guessed y_p as many times as needed in order to plug in guessed y_p, y_p', y_p'' into problem statement.
- plug in y_p, y_p', y_p'' into problem statement with $g(t)$ on right hand side
- compare left & right hand sides of eqn to find coefficients of guessed y_p.
- check that the guessed y_p (with coefficients) are valid

NOT applicable to any general SLDE.

YES, y_p depends on the corresponding homogeneous solution in that if a term is repeated, the guessed y_p is multiplied by t until it is in a form not represented in the homogeneous solution.

b. $y'' - 4y' + 4y = g(t)$

C.E. $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

roots: 2,2

$y_{homogeneous} = C_1 e^{2t} + C_2 t e^{2t}$

1. $g(t) = t\sin(t)e^t$

$\therefore y_p(t) = [(A_0 + A_1 t)\cos(t) + (B_0 + B_1 t)\sin(t)]e^t$

Problem 2

b. 2. $g(t) = \sin(t) + te^{2t}$

$\therefore y_p(t) = A_1\cos(t) + A_2\sin(t) + t^2(A_3 + A_4 t)e^{2t}$

3. $g(t) = t + \cos(t) + e^{2t}$

$\therefore y_p(t) = (A_0 + A_1 t) + A_2\cos(t) + A_3\sin(t) + A_4 t^2 e^{2t}$

complete general solution:

$y_p(t) = A_0 + A_1 t + A_2\cos(t) + A_3\sin(t) + A_4 t^2 e^{2t}$

$y_p'(t) = A_1 - A_2\sin(t) + A_3\cos(t) + 2A_4 te^{2t} + 2A_4 t^2 e^{2t}$

$y_p''(t) = -A_2\cos(t) - A_3\sin(t) + 2A_4 e^{2t} + 4A_4 te^{2t} + 4A_4 te^{2t}$
$\quad + 4A_4 t^2 e^{2t}$

plug into $y'' - 4y' + 4y = g(t)$. & rearrange for comparison

$y_p'' - 4y_p' + 4y_p = t + \cos(t) + e^{2t}$

$-A_2\cos(t) - A_3\sin(t) + 2A_4 e^{2t} + 4A_4 te^{2t} + 4A_4 te^{2t} + 4A_4 t^2 e^{2t}$
$-4(A_1 - A_2\sin(t) + A_3\cos(t) + 2A_4 te^{2t} + 2A_4 t^2 e^{2t})$
$+4(A_0 + A_1 t + A_2\cos(t) + A_3\sin(t) + A_4 t^2 e^{2t})$
$= t + \cos(t) + e^{2t}$

$(4A_0 - 4A_1) + (3A_2 - 4A_3)\cos(t) + (3A_3 + 4A_2)\sin(t)$
$+(4A_1)t + (2A_4)e^{2t} + \underbrace{(8A_4 - 8A_4)}_{\to 0}te^{2t} + \underbrace{(8A_4 - 8A_4)}_{\to 0}t^2e^{2t}$
$= t + \cos(t) + e^{2t}$

$(4A_0 - 4A_1) + (3A_2 - 4A_3)\cos(t) + (3A_3 + 4A_2)\sin(t) + (4A_1)t$
$+ (2A_4)e^{2t} = t + \cos(t) + e^{2t}$

Problem 2

b. <u>complete general solution</u>

compare:

$$\left.\begin{array}{lr} \text{constants}: & 4A_0 - 4A_1 = 0 \\ t: & 4A_1 = 1 \\ \cos(t): & 3A_2 - 4A_3 = 1 \\ \sin(t): & 4A_2 + 3A_3 = 0 \\ e^{2t}: & 2A_4 = 1 \end{array}\right\} \quad \therefore \begin{array}{l} A_0 = \frac{1}{4} \\ A_1 = \frac{1}{4} \\ A_2 = \frac{3}{25} \\ A_3 = -\frac{4}{25} \\ A_4 = \frac{1}{2} \end{array}$$

general: $y_p + y_h = y_g$

$$\therefore y_g = C_1 e^{2t} + C_2 t e^{2t} + \frac{1}{4} + \frac{1}{4} t + \frac{3}{25}\cos(t) - \frac{4}{25}\sin(t) + \frac{1}{2} t^2 e^{2t}$$

Problem 3 (4+4+6+6)

Consider a linear, $n-th$ order differential equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = g(t), \quad y^{(n)} = \frac{d^n y}{dx^n}$$

a. Explain the procedures leading to the complete solution of the homogenous equation $L[y] = 0$

b. Does the form of the particular solution $y_p(t)$ of $L[y] = g(t)$ depend on the coefficients $a_0, a_1, \ldots . a_n$ and the form of $g(t)$? If so, explain the nature of such dependency.

c. For the fourth order equation $L[y] = y^{(4)} + 3y'' - 4y = g(t) = t + t^2 e^t + 3t\sin(2t)$, derive the general solution of $L[y] = 0$, and

d. Using result in (c), write down the form of the particular solution y_p of $L[y] = g(t)$. Explain your form of y_p. **Do not solve for the unknown constants in your formulation.**

Solution

(a) Using the trial solution $y = e^{\lambda t}$ in $L[y] = 0$, λ satisfies the n-th degree equ^{n} $a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$.
This equ^{n} yields n-sol^{n} of $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$. Hence the solution can be written as $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$.
These fundamental solutions should be modified if the roots are repeated.

(b) Yes since the form of y_p depends on $g(t)$ as well as the roots $\lambda_1, \lambda_2, \ldots \lambda_n$, which depend on $a_0, a_1, \ldots a_n$.

(c) Using $y \sim e^{\lambda t}$, λ satisfies $\lambda^4 + 3\lambda^2 - 4 = 0$ in $L[y] = 0$

$$\lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = 1, -4 \Rightarrow \lambda = 1, -1, -2i, 2i$$

Fundamental Sol^{n}: $e^t, e^{-t}, \sin 2t, \cos 2t$

General Sol^{n} of $L[y] = 0$, $\underline{y = c_1 e^t + c_2 e^{-t} + c_3 \sin 2t + c_4 \cos 2t}$

(d) When $g(t) = t + t^2 e^t + 3t \sin(2t)$.

$$g(t) = (A_1 + A_2 t) + te^t(B_1 + B_2 t + B_3 t^2)e^t + \\ + t[(C_1 + C_2 t)\sin 2t + (D_1 + D_2 t)\cos 2t]$$

Since e^t and $\sin 2t$ are fundamental Sol^{n} of $L[y] = 0$