MECHANICAL & AEROSPACE ENGINEERING DEPARTMENT
UNIVERSITY OF CALIFORNIA, LOS ANGELES

MAE 182A

MATHEMATICS OF ENGINEERING

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INSTRUCTOR

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FINAL EXAMINATION

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INSTRUCTIONS: SHOW ALL CALCULATIONS ON THESE PAGES.
ATTACH ADDITIONAL PAGES AS NECESSARY.

NAME
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Your Student ID ____

Problem 1 **(4+8+3)**

Consider the first order, differential equation of the form $y' = f(y)$.

1. Define the following terms;
   a. critical or equilibrium solution
   b. asymptotically stable solution
   c. semistable solution
   d. unstable solution

2. If $f(y) = ay - b\sqrt{y},\ y \geq 0,\ a > 0,\ b > 0,\ y(0) = y_0 \geq 0$, determine the equilibrium points,and classify each one as asymptotically stable, unstable or semistable.

3. Sketch (do not solve the differential equation) the nature of solution for various y_0 near the equilibrium points.

1.
   a. If $y = y_0$ is a solution of $f(y) = 0$, then $y = y_0$ is an equilibrium solution of the differential equation. If the initial condition is $y(0) = y_0$, then $y(t) = y_0$ for all t.
   b. If the initial condition is $y(0) = y_1$ where y_1 is in the neighborhood of y_0, and the solution $y(t)$ approaches to y_0 i.e., $\lim_{t\to\infty} y(t) = y_0$, then $y = y_0$ is said to be asymptotically stable.
   c. If the initial condition is $y(0) = y_1$ where y_1 is in the neighborhood of y_0, and the solution $y(t)$ approaches to y_0 i.e., $\lim_{t\to\infty} y(t) = y_0$, when $y_1 > y_0$, but moves away from y_0, when $y_1 < y_0$ or vice versa, then $y = y_0$ is said to be a semistable solution
   d. If the initial condition is $y(0) = y_1$ where y_1 is in the neighborhood of y_0, and the solution $y(t)$ moves away from y_0 when time increases, then $y = y_0$ is said to be an unstable solution.

The nature of equilibrium solution can be derived from the nature of $f(y)$ in the neighborhood of $y = y_0$.

2. Here $f(y) = ay - b\sqrt{y} = \sqrt{y}(a\sqrt{y} - b)$. Since the equilibrium solution are the roots of $f(y) = 0$, we have two equilibrium solutions $y = y_0 = 0,\ \dfrac{b^2}{a^2}$.

Near $y = 0, y > 0$, $f(y)$ is negative, and hence the solution return to the stable state $y = 0$. Thus, $y = 0$ is a stable equilibrium point.

Near the initial value y_1 is near $y = \frac{b^2}{a^2} = y_2$, then $f(y) = \begin{cases} > 0 \text{ if } y_1 > y_2 \\ < 0 \text{ if } y_1 < y_2 \end{cases}$

Hence $y = y_2$ is a semistable equilibrium state.

Problem 2 (15)

The displacement of a mass-spring-dashpot model of a forced, dynamic system is governed by the differential equation $m\ddot{x}+c\dot{x}+kx=F(t)$.

If $m=2kg,\ k=3\frac{N}{m},\ c=1\frac{N.s}{m},\ F(t)=4\sin 3t-3\cos 3t$, **derive the steady state response in** the amplitude-phase shift form $R\sin(\omega t-\delta)$.

Do not use any formula.

Solution

The solution of the differential equation can be written as $u(t)=u_c(t)+u_p(t)$ where $u_c(t)$ is the solution to the homogeneous equation ($F(t)=0$), and $u_p(t)$ is the particular solution. In the presence of damping, $u_c(t)$ approaches zero as time increases, ane hence the steady state solution $u_s(t)$ is equal to $u_p(t)$.

$u_p(t)$ can be derived using the method of undetermined constants. Since the loading $F(t)=4\sin 3t-3\cos 3t$, $u_p(t)$ can be written as

$$u_p(t)=a\cos\omega t+b\sin\omega t,\ \omega=3$$

$$u_p{}'(t)=\omega\left(-a\sin\omega t+b\cos\omega t\right)\quad .$$

$$u_p{}''(t)=-\omega^2\left(a\cos\omega t+b\sin\omega t\right)$$

Substituting in the differential equation, we have

$$-2\omega^2\left(a\cos\omega t+b\sin\omega t\right)-\omega\left(a\sin\omega t-b\cos\omega t\right)+$$
$$+3\left(a\cos\omega t+b\sin\omega t\right)=4\sin\omega t-3\cos\omega t$$

Equating coefficients of $\cos\omega t$ and $\sin\omega t$ we have,

$$-2\omega^2 a+\omega b+3a=-3$$
$$-2\omega^2 b-\omega a+3b=4$$

Solving for a and b, we have

Hnce the steady state solution is

Problem 3 (5+12+3)

4. Show that $x = 0$ is a regular singular point of the differential equation
$$3x^2y'' + 2xy' + x^2y = 0$$
5. Find two fundamental series solutions of the above equation **upto three nonzero terms**.
6. Are these solutions bounded as $x \to 0$

Note: No formulae needed. Directly substitute correct form of solution and equate coefficients of like terms.

Solution:

Here $xp(x) = \dfrac{2}{3}$, and $x^2q(x) = \dfrac{x^2}{3}$, which are both analytic at $x = 0$. Thus at $x = 0$,

$P(x) = 3x^2 = 0$, and $\lim_{x\to 0}[xp(x)] = \dfrac{2}{3} = \mathit{finite}$ and $\lim_{x\to 0}\left[x^2q(x)\right] = 0 = \mathit{finite}$. **Hence**

$x = 0$ **is a regular singular point**. Set

$y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$3\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} = 0\,.$$

It follows that

$$a_0[3r(r-1)+2r]x^r + a_1[3(r+1)r + 2(r+1)]x^{r+1} +$$
$$+\sum_{n=2}^{\infty}[3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, and

$$a_n = \frac{-a_{n-2}}{(r+n)[3(r+n)-1]}\,,\quad n = 2, 3, \cdots\,.$$

It immediately follows that the *odd* coefficients are equal to *zero*. For $r = 1/3$,

$$a_n = \frac{-a_{n-2}}{n(1+3n)}\,,\quad n = 2, 3, \cdots\,.$$

So for $k = 1, 2, \cdots$,

Problem 3: Solution Continued….

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k+1)} = \frac{(-1)^k a_0}{2^k\, k!\, 7 \cdot 13 \cdots (6k+1)}\,.$$

For $r = 0$,

$$a_n = \frac{-a_{n-2}}{n(3n-1)}\,, \quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k\, k!\, 5 \cdot 11 \cdots (6k-1)}\,.$$

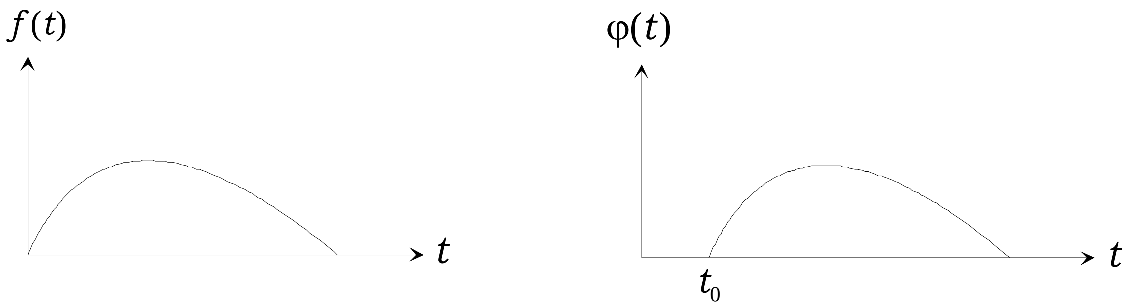
The two linearly independent solutions are

$$y_1(x) = x^{1/3}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 7 \cdot 13 \cdots (6k+1)} \left(\frac{x^2}{2}\right)^k\right]$$

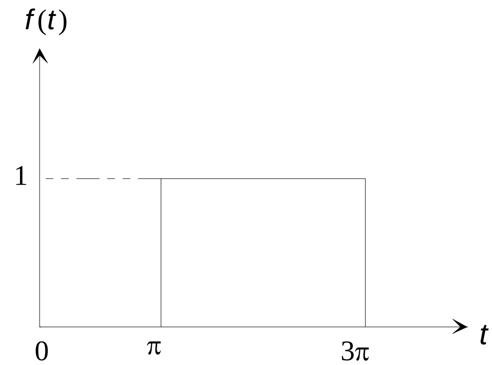
$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 5 \cdot 11 \cdots (6k-1)} \left(\frac{x^2}{2}\right)^k.$$

Problem 4 **(4+3+8)**

1. Using the definition of Laplace transform, derive the relation between the Laplace transforms of $f(t)$ and $\varphi(t)$ where $\varphi(t)$ is obtained by a translation of $f(t)$ a distance t_0 as shown below.



2. Recalling the definition of a unit step function $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$, express the load function $f(t)$ as shown in terms of $u_c(t)$



3. Using the result derived in section 2 and Laplace transform, find the complete solution of

$$\ddot{y} + 4y = f(t),\ y(0) = 0 = \dot{y}(0)$$

Problem 5 (20)

Find the general solution of the system of linear, first order differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t, \ \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

using the following steps.

- First solve the homogeneous problem using eigen values and eigen vectors
- Solve the nonhomogeneous problem using the method of undetermined constants.

7. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of *undetermined coefficients*. Set $\mathbf{v} = \mathbf{a}\, e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

In scalar form, after canceling the exponential, we have

$$\begin{aligned} a_1 &= a_1 + a_2 + 2 \\ a_2 &= 4a_1 + a_2 - 1\,, \end{aligned}$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t,$$

so that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} e^t \\ -8e^t \end{pmatrix}.$$

8. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Problem 6 **(10+5)**

Consider the two-point boundary value problem defined by

$$y'' + \pi^2 y = a - \cos \pi x,\ 0 \le x \le 1,\ y(0) = 0 = y(1)$$

1. **1.** Find the eigen values and eigen functions of the corresponding Sturm-Liouville boundary value (SLBV) problem.
2. **2.** Does the solution exist for all values of a? If not, find the value of a for which solution exists.
3. **3.** Find the solution of the differential equation for the value of a found in the section 1

The differential equation can be written as

$$L[y] = -y'' = \pi^2 y - a + \cos \pi x,\ 0 \le x \le 1,\ y(0) = 0 = y(1) \qquad (1)$$

1. Associated Sturm-Liouville two-point boundary value problem is

$$L[y] = -y'' = \lambda y,\ 0 \le x \le 1,\ y(0) = 0 = y(1) \quad (2)$$

where λ is the eigen value.

Set $\lambda = \mu^2 \ge 0$, then the general solution is $y(x) = c_1 \cos \mu x + c_2 \sin \mu x$. Boundary conditions (BC) give $c_1 = 0$ and $c_2 \sin \mu = 0$. Since $c_2 \ne 0$ as that will make $y(x) \equiv 0$, we get the characteristic equation $\sin \mu = 0 = \sin n\pi,\ n = 1, 2, 3...$, Thus the eigen values are $\lambda_n = n^2\pi^2,\ n = 1, 2, 3....$, and the eigen functions are $y = \varphi_n(x) = k_n \sin(n\pi x)$. Normalizing the eigen functions using the normalization condition $\int_0^1 \varphi_n^2(x)dx = 1$, we get $k_n = \sqrt{2}$. Hemce the normalized eigen functions are

$$\varphi_n(x) = \overline{\varphi}_n(x) = \sqrt{2}\sin(n\pi x) \qquad (3)$$

2. Solution of the nonhomogeneous problem

Writing the differential equation as $-y'' = \eta y + f(x),\ \eta{=}\pi^2, f(x) = -a + \cos(\pi x), 0 \le x \le 1,\ y(0) = 0 = y(1)$, the solution $y(x)$ and $f(x)$ can be expanded as

$$y(x) = \sum_{n=1}^{\infty} a_n \overline{\varphi}_n(x) \quad , \ f(x) = \sum_{n=1}^{\infty} c_n \overline{\varphi}_n(x) \ \text{ where } c_n = \int_0^1 \text{f(x)}\overline{\varphi}_n(x)dx .$$

Then $y(x)$ satisfies the BC of the problem as all $\overline{\varphi}_n(x)$ satisfies the BC. Substituting in the differential equation, and using $L[\overline{\varphi}_n(x)] = \lambda_n \overline{\varphi}_n(x)$, we have, after equating the coefficients of $\overline{\varphi}_n(x)$, $\lambda_n a_n = \eta a_n + c_n$ or $a_n(\lambda_n - \eta) = c_n$.

If $\lambda_n \neq \eta$, $a_n = \dfrac{c_n}{\lambda_n - \eta}$.

If $\lambda_n = \eta$, and $c_n = 0$, a_n is arbitrary. If $\lambda_n = \eta$, and $c_n \neq 0$, then the problem has no solution.

In this case $\lambda_1 = \pi^2 = \eta$, hence we need

$c_1 = \int_0^1 f(x)\sin(\pi x)\,dx = \int_0^1 [-a + \cos\pi x]\sin(\pi x)\,dx = 0$. This gives $a = 0$

3. Final solution of the inhomogeneous problem can be written as

$$y(x) = \sum_{n=1}^{\infty} c_n \overline{\varphi}_n(x) \quad = 2\sum_{n=2}^{\infty} \frac{\int_0^1 \cos\pi x \sin(n\pi x)\,dx}{(n^2-1)\pi^2} \sin(\mathrm{n}\pi\mathrm{x}) \quad + c\sin\pi x$$

$$\text{or } y(x) = 4\sum_{n=2}^{\infty} \frac{n\sin(\mathrm{n}\pi\mathrm{x})}{(n^2-1)^2\pi^3} \quad + c\sin\pi x$$

where c is arbitrary.

Solving the nonhomgeneos problem directly suing methods of Chapter 3, we get

$$y(x) = -\frac{x}{2\pi}\sin\pi x + c\sin\pi x$$

These two solutions are identical as it can be shown that the difference between these two solutions is proportional to $\sin\pi x$.

Formulae Page

Problem 4

If $L\left[f(t)\right]=F(s)$, then $L^{-1}\left[sF(s)\right]=\dfrac{df}{dt}+f(0)\delta(t)$ where L and L^{-1} are the forward and inverse Laplace transform operators, respectively.

$$L\left[\delta(t)\right]=1\ ;\ L\left[u_c(t)\right]=\frac{e^{-cs}}{s}$$

$$L^{-1}\left[\frac{1}{(s-s_1)(s-s_2)}\right]=\frac{e^{s_1t}-e^{s_2t}}{(s_1-s_2)}\ ;\ L^{-1}\left[\frac{1}{(s+a)^2}\right]=te^{-at}$$

If $L\left[f(t)\right]=F(s)$, then

$$L\left[\dot{f}(t)\right]=sF(s)-f(0)$$

$$L\left[\ddot{f}(t)\right]=s^2F(s)-sf(0)-\dot{f}(0)$$

where L is the Laplace transform operator.

More formulae will be added later.