MECHANICAL & AEROSPACE ENGINEERING DEPARTMENT
UNIVERSITY OF CALIFORNIA, LOS ANGELES

MAE 182A

MATHEMATICS OF ENGINEERING

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INSTRUCTOR

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FINAL EXAMINATION

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INSTRUCTIONS: SHOW ALL CALCULATIONS ON THESE PAGES.
ATTACH ADDITIONAL PAGES AS NECESSARY.

NAME ____

Your Student ID ____

Problem 1 (5+5+10)

Consider the nonhomogeneous second order differential equation (SDE) with constant coefficients $ay'' + by' + cy = f(t)$ where a, b, c are constants.

1. Method of undetermined coefficients is used to find the particular solution of the problem. Explain how this method works.
2. Is this method applicable to any general form of $f(t)$? If not, why not.
3. Derive the complete solution of $y'' + 2y' + 5y = 4e^{-t}\cos(2t),\ y(0) = 1,\ y'(0) = 0$ using the method of undetermined coefficients.

Problem 2 (5+12+3)

1. Show that $x = 0$ is a regular singular point of the differential equation
$$2x^2 y'' + 3xy' + (2x^2 - 1)y = 0$$
2. Find two fundamental series solutions of the above equation upto three terms.
3. Are these solutions bounded as $x \to 0$

Problem 3 **(5+5+8+2)**

The displacement $y(t)$ of a dynamic system is governed by the differential equation

$$\ddot{y} + 4\dot{y} + 4y = g(t), \ \mathrm{y}(0) = 0 = \dot{y}(0)$$

Let $y(t) = y_1(t)$ when $g(t) = \delta(t)$, and $\mathrm{y}(t) = y_2(t)$ when $g(t) = u_0(t)$

1. Derive the mathematical equation relating the two displacements $y_1(t)$ and $y_2(t)$ **(Hint: See the attached formulae page)**

2. When $g(t) = \delta(t)$, reduce the above **nonhomogeneous** differential equation with **zero** boundary conditions to a **homogeneous** differential equation with a **non-zero** boundary condition. Write down the homogeneous differential equation and corresponding boundary conditions. **(Hint: See the attached formulae page)**

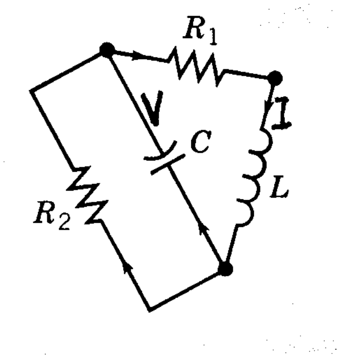
3. Using the Laplace transform, derive the complete solution of the differential equation

$$\ddot{y} + 4\dot{y} + 4y = te^{-t}, \ \mathrm{y}(0) = 0 = \dot{y}(0)$$

Do not use the formula on the attached formulae page. *You may leave your solution in integral forms.*

4. What is the steady state displacement?

Problem 4 (5+10+5)



1. For the electric circuit shown, show that the current $I(t)$ through the inductance L and voltage $V(t)$ across the capacitance C , satisfy the coupled first order differential equations

$$\begin{pmatrix} \dot{I} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}$$

Use the Kirchoff's laws on the attached formulae page.

2. If $R_1 = 4$ ohms, $R_2 = 4$ ohms, $C = \frac{1}{2}$ farad, and $L = 8$ henrys, determine the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$.

3. Using the $\Phi-$matrix, solve the differential equation satisfying the initial conditions $I(0) = 2$ amperes, $V(0) = 3$ volts.

Problem 5 (4+8+8)

Consider the heat conduction problem of finding the temperature $u(x,t)$ in a uniform rod of length ℓ with **one end insulated and other end held at zero temperature.** The differential equation satisfied by $u(x,t)$ is given by the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa^2}\frac{\partial u}{\partial t}\ , \ 0 \le x \le \ell \text{ and}$$

$$\text{the boundary conditions are } u(0,t) = 0 = \frac{\partial}{\partial x}\left[u(\ell,t)\right]$$

where κ^2 is the thermal diffusivity. The initial temeparture along the rod is given by

$$u(x,0) = \sin\left(\frac{\pi x}{\ell}\right)\ , \ 0 \le x \le \ell$$

1. Using a variable separable form $u(x,t) = X(x)T(t)$, reduce the problem to a Sturm-Liouville boundary value (SLBV) problem.
2. Determine eigen values and eigen functions of the SLBV problem.
3. Derive the complete solution of the transient temperature field $u(x,t)$ in terms of the normalized eigen functions and time t.

Formulae Page

1. Kirchoff's Laws (Q =charge, $I=\dfrac{dQ}{dt}$ *=current)*

$V_R = RI$ (Voltage drop across resistor R)

$V_c = \dfrac{q}{C}$ (Voltage drop across capacitor C)

$V_L = L\dfrac{dI}{dt}$ (Voltage drop across inductor L)

In a closed circuit, sum of the voltage drops is zero.
At any node, the **mathematical** sum of the currents is zero.

2. The general solution of $m\ddot{u}+c\dot{u}+ku=g(t),\ u(0)=u_0$, $\dot{u}(0)=\dot{u}_0$ is given by

$$u(t)=u_0\left[\frac{d\varphi}{dt}+\left(\frac{c}{m}\right)\varphi\right]+\dot{u}_0\varphi(t)+\frac{1}{m}\int_0^t\varphi(t-\tau)g(\tau)d\tau$$

Where $\varphi(t)=\dfrac{e^{-\left(\frac{c}{2m}\right)t}\sin(\mu t)}{2\mu}$, $\mu=\sqrt{\dfrac{k}{m}-\dfrac{c^2}{4m^2}}$

3. If $L\left[f(t)\right]=F(s)$, then $L^{-1}\left[sF(s)\right]=\dfrac{df}{dt}$ where L and L^{-1} are the forward and inverse Laplace transform operators, respectively.

4.
$$L\left[\delta(t)\right]=1\ ;\ L\left[u_c(t)\right]=\frac{e^{-cs}}{s}$$
$$L\left[\frac{1}{(s-s_1)(s-s_2)}\right]=\frac{e^{s_1t}-e^{s_2t}}{(s_1-s_2)}\ ;\ L\left[\frac{1}{(s+a)^2}\right]=te^{-at}$$

5.
$$L\left[\dot{f}(t)\right]=sF(s)-f(0)$$
$$L\left[\ddot{f}(t)\right]=s^2F(s)-sf(0)-\dot{f}(0)$$