Reliability Engineering Fundamentals

Fall 2016

Midterm Exam, 2 hrs. Open Book, No Internet Access (8 questions)

SOLUTION

Problem 1 (10%)

Consider three independent events E_1, E_2, and E_3, with probabilities $P(E_1)=0.9$, $P(E_2)=0.3$, and $P(E_3)=0.5$, respectively.

(a) What is the probability of the event $\{(E_1 \cup E_2) \cap E_3\}$?
(b) What is the probability of the event $\{E_1 \cup (E_2 \cap E_3)\}$?

SOLUTION

(a) What is the probability of the event $\{(E_1 \cup E_2) \cap E_3\}$?

$$P\{(E_1 \cup E_2) \cap E_3\} = P(E_1 \cup E_2) \cdot P(E_3) = [P(E_1) + P(E_2) - P(E_1) \cdot P(E_2)] \cdot P(E_3) =$$
$$= [0.9 + 0.3 - 0.9 \cdot 0.3] \cdot 0.5 = 0.465$$

(b) What is the probability of the event $\{E_1 \cup (E_2 \cap E_3)\}$?

$$P\{E_1 \cup (E_2 \cap E_3)\} = P(E_1) + P(E_2 \cap E_3) - P(E_1) \cdot P(E_2 \cap E_3) =$$
$$= P(E_1) + P(E_2) \cdot P(E_3) - P(E_1) \cdot P(E_2) \cdot P(E_3) =$$
$$= 0.9 + 0.3 \cdot 0.5 - 0.9 \cdot 0.3 \cdot 0.5 = 0.915$$

Problem 2 (6%)

A random variable X follows a Log-Normal distribution. The 5^{th} and 95^{th} percentiles of the distribution are 0.001 and 0.01 respectively.

(a) Find the logarithmic standard deviation σ
(b) Calculate the median of X
(c) Calculate the mean of X

Note: the value of the Standard Normal Variable, Z, at 5% and 95% cumulative probability are -1.645 and +1.645, respectively.

SOLUTION

(a) Find the logarithmic standard deviation σ.

From the given data, we can write the following system of equations:

$$\begin{cases} z_5 = \dfrac{\ln(x_5) - \ln(\mu)}{\sigma} \\ z_{95} = \dfrac{\ln(x_{95}) - \ln(\mu)}{\sigma} \end{cases} \Rightarrow \begin{cases} -1.645 = \dfrac{\ln(0.001) - \ln(\mu)}{\sigma} \\ 1.645 = \dfrac{\ln(0.01) - \ln(\mu)}{\sigma} \end{cases}$$

Solving it for σ gives:

$$\sigma = 0.7$$

(b) Calculate the median of X.

Also, solving for μ gives:

$$\mu = 3.16\text{x}10^{-3}$$

(c) Calculate the mean of X.

The mean of X is given by:

$$\mathbb{E}(X) = \mu e^{\frac{\sigma^2}{2}} = 4.02 \times 10^{-3}$$

Problem 3 (4%)

During the peak hours of 11 am to 1 pm customers arrive at a fast food restaurant at the rate of 3 per minute.

(a) What is the expected (average) number of people arriving during the peak time interval?
(b) What is the probability that 50 customers will visit the place during peak time?

SOLUTION

(a) What is the expected (average) number of people arriving during the peak time interval?

We are trying to model the number of times an event occurs in an interval of time. The Poisson distribution is an appropriate choice. For the T=2h interval with a rate of λ=3 visits per minute, we can calculate the expected number of people as or this interval of time:

$$\mu = T \cdot \lambda = 2\, h \cdot \frac{60\, min}{1\, h} \cdot 3 \frac{people}{min} = 360\, people$$

(b) What is the probability that 50 customers will visit the place during peak time?

Using the Poisson distribution already defined, we get:

$$P(X = 50, \mu = 360) = \frac{e^{-360} \cdot 360^{50}}{50!}$$

Problem 4 (15%)

Show that the mean time to failure (MTTF) can be calculated from

$$\text{MTTF} = \int_0^\infty \text{R}(\text{t})\, \text{dt}$$

where R(t) is the reliability at time t.

SOLUTION

MTTF is defined as the expected value of time to failure. Mathematically, MTTF is given by:

$$MTTF = \int_0^\infty t f(t) dt$$

Next, we use the following expression:

$$f(t) = -\frac{dR(t)}{dt}$$

to obtain:

$$MTTF = \int_0^\infty t\left[-\frac{dR(t)}{dt}\right]dt$$

We integrate by parts and apply l'Hospital's rule to obtain the required form of the expression:

$$MTTF = \int_0^\infty R(t)dt$$

Problem 5 (15%)

A battery on store shelf deteriorates (loses all charge) at the rate of 1/10,000 hrs. A customer buys a pack of 4 batteries that has been on the shelf for 2 years and installs them in his videogame controller. The controller requires at least 3 batteries to operate. What is the probability that the controller would not have adequate power supply?

SOLUTION

The failure rate is $\lambda = \frac{1}{10000}$ hrs. The reliability of the battery for t unit time is $R(t) = e^{-\lambda t}$.

The probability of a battery to operate is hence

$t = 2 \times 365 \times 24 = 17520,\ R(17520) = e^{\frac{-17520}{10000}} \simeq 0.17$.

The probability of not having adequate power, F, is equal to the complement probability of operating at least 3 batteries, that is

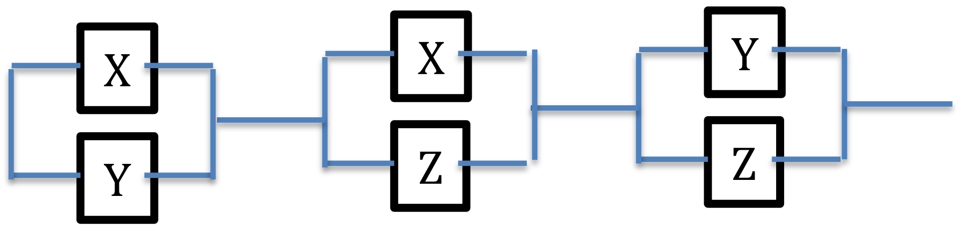
$F = 1 - (4 \times R^3 \times (1 - R)) - R^4 \simeq 0.98$.

Problem 6 (10%)

A system is made of 3 identical components X, Y, and Z. The system fails if any two of the three components fail. Draw a Reliability Block Diagram of the system.

SOLUTION

The reliability block diagram is a parallel-series configuration as follows. Since the components are identical, the position of X, Y and Z can be exchanged.



Problem 7 (10%)

A logic circuit is known to have an increasing hazard rate as follows:
$(t) = 0.22t$, where t is time (measured in years).

(a) What is the reliability of this logic circuit at the end of one year of operation?
(b) What is the probability that the circuit fails between 2 and 4 months of operation?

SOLUTION

a) The failure rate is $\lambda(t) = 0.22t$. The reliability at the end of t unit time is

$$R(t) = \mathrm{e}^{-\int_0^t \lambda(t)\,dt} = \mathrm{e}^{-\int_0^t 22t\,dt} = e^{-0.11t^2}.$$

Therefore, the reliability of this logic circuit at the end of one year is
$R(1) = e^{-0.11} \simeq 0.895$.

b) The probability of failure between 2 and 4 months is

Prob (failure between 2 and 4 months) = F (4 months) – F (2 months)

= {1-R (4 months)} – {1-R (2 months) }=

$$\left(1-R\left(\frac{4}{12}\right)\right)-\left(1-R\left(\frac{2}{12}\right)\right)=R\left(\frac{1}{6}\right)-R\left(\frac{1}{3}\right)=e^{\frac{-0.11}{36}}-e^{\frac{-0.11}{9}}\simeq 0.009$$

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Problem 8 (30%)

A simple system is made of two components in series arrangement. Twenty of these systems are tested for 1000 hours each. The test data (failure times in hours) for the two components are shown in the following table.

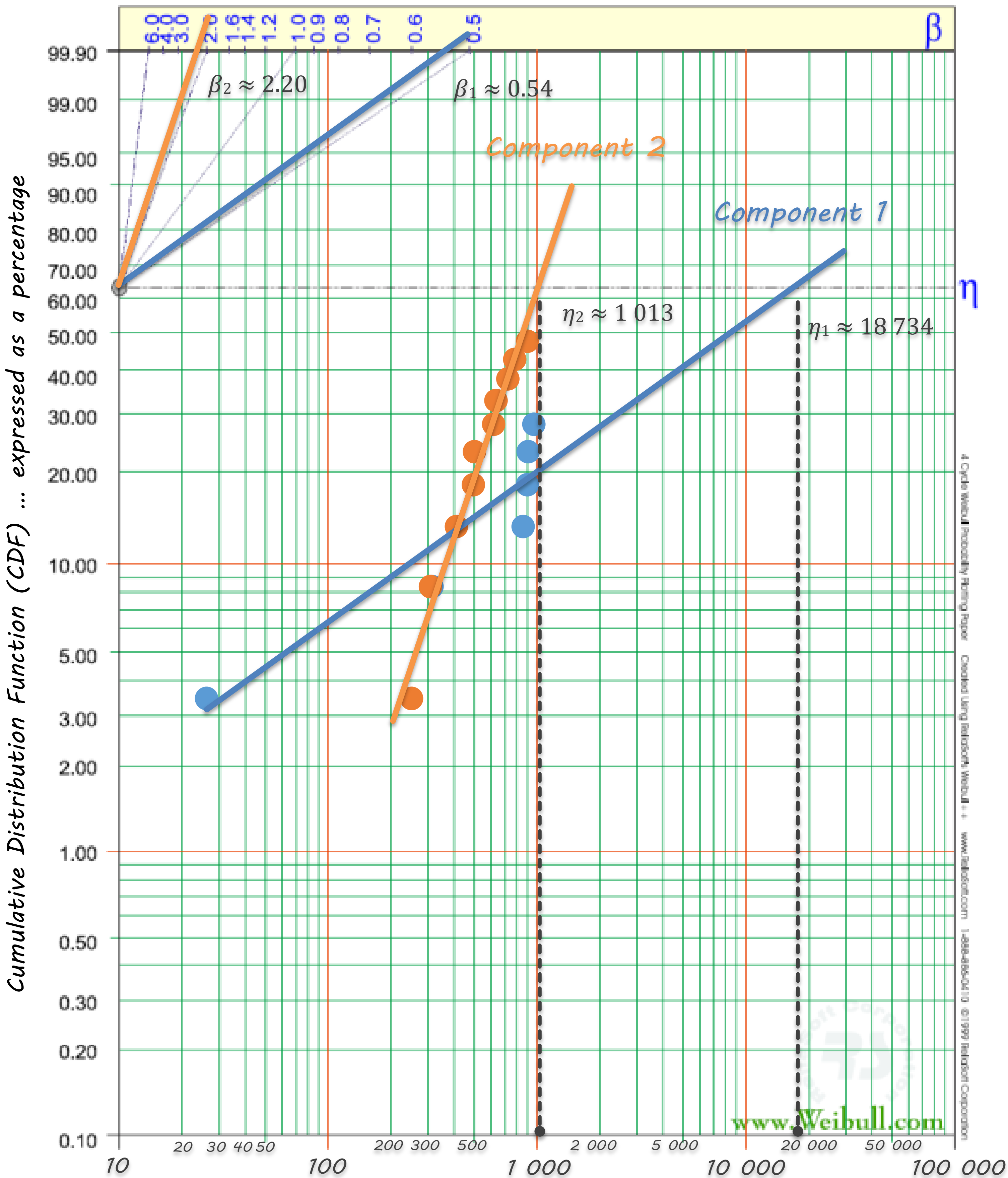
(a) Plot the times to failure for the two components on the provided Weibull probability plotting paper and estimate the parameters for the Weibull distribution that best fits each component's data.
(b) Calculate system failure rate at 1000 hours based on results from part (a)
(c) What would be the system failure rate at 1000 hours if we assume constant failure rate for both components?

	Component 1	Component 2
1	26	250
2	314	308
3	858	411
4	899	495
5	905	501
6	966	620
7		635
8		725
9		781
10		900
Number of items still functional at the completion of the test (1000 hours):	**14**	**10**

SOLUTION

(a) The Weibull parameter estimates are as follows (note plot on subsequent page):

Component	β	η (hours)
1	0.54	18 734
2	2.20	1 013


β
6.0
4.0
3.0
2.0
1.6
1.4
1.2
1.0
0.9
0.8
0.7
0.6
0.5
$\beta_2 \approx 2.20$
$\beta_1 \approx 0.54$
Component 2
Component 1
η
$\eta_2 \approx 1\ 013$
$\eta_1 \approx 18\ 734$
99.90
99.00
95.00
90.00
80.00
70.00
60.00
50.00
40.00
30.00
20.00
10.00
5.00
3.00
2.00
1.00
0.50
0.30
0.20
0.10
Cumulative Distribution Function (CDF) ... expressed as a percentage
10
20
30
40
50
100
200
300
500
1 000
2 000
5 000
10 000
20 000
50 000
100 000
www.Weibull.com

(b) Calculate system failure rate at 1000 hours based on results from part (a)

The failure rate of a series system at any time is the sum of the failure rates as follows:

System failure rate is defined as:

$$\lambda_{system}(t) = \frac{f_{system}(t)}{R_{system}(t)} = \lim_{\Delta t \to 0} \frac{F_{system}(t + \Delta t) - F_{system}(t)}{\Delta t \cdot R_{system}(t)}$$

This is another way of saying that the hazard rate of the system is the probability that the system will fail between now (time t) and some small period of time in the future (time t):

$$\lambda_{system}(t) = \lim_{\Delta t \to 0} \frac{\Pr(system\ fails\ between\ t\ and\ t + \Delta t)}{\Delta t}$$

For a two component series system made up of components a and b (where we define events A and B to be the failure of components a and b respectively):

$$\lambda_{system}(t) = \lim_{\Delta t \to 0} \frac{\Pr(A \cup B)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pr(A) + \Pr(B) - \Pr(A \cap B)}{\Delta t}$$

We know that as $\Delta t \to 0$, $\Pr(A \cap B)$ becomes insignificantly small in comparison with $\Pr(A) + \Pr(B)$ … which is the assumption upon which rare event calculation is based.

$$\lambda_{system}(t) = \lim_{\Delta t \to 0} \frac{\Pr(A \cup B)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pr(A) + \Pr(B)}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{\Pr(A)}{\Delta t} + \frac{\Pr(B)}{\Delta t}\right]$$

By the definitions of events A and B, we can now insert component failure rate.

$$\lim_{\Delta t \to 0} \frac{\Pr(A)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pr(component\ a\ fails\ between\ t\ and\ t + \Delta t)}{\Delta t} = \lambda_a(t)$$

Therefore,

$$\lambda_{system}(t) = \lim_{\Delta t \to 0} \left[\frac{\Pr(A)}{\Delta t} + \frac{\Pr(B)}{\Delta t}\right] = \lambda_a(t) + \lambda_b(t)$$

Using the parameters from answer (a), and the expressions for the Weibull distribution:

$$\lambda_{component}(t) = \frac{f_{component}(t)}{R_{component}(t)} = \frac{\left[\frac{\beta t^{\beta-1}}{\eta^{\beta}} e^{-\left(\frac{t}{\eta}\right)^{\beta}}\right]}{e^{-\left(\frac{t}{\eta}\right)^{\beta}}} = \frac{\beta t^{\beta-1}}{\eta^{\beta}}$$

Using the parameter estimates above:

$$\lambda_{component\ 1}(1000) = \frac{\beta_1 t^{\beta_1 - 1}}{\eta_1^{\beta_1}} = \frac{0.54 \cdot t^{0.54-1}}{18734^{0.54}}$$

$$= 0.000245\ failures\ per\ hour$$

$$\lambda_{component\ 2}(1000) = \frac{\beta_2 t^{\beta_2 - 1}}{\eta_2^{\beta_2}} = \frac{2.20 \cdot t^{2.20-1}}{1013^{2.20}}$$

$$= 0.001585\ failures\ per\ hour$$

Therefore, the system hazard rate at 1000 hours is:

$$\lambda_{system}(1000) = \lambda_{component\ 1}(1000) + \lambda_{component\ 2}(1000)$$

$$= 0.00183\ failures\ per\ hour$$

(c) What would be the system failure rate at 1000 hours if we assume constant failure rate for both components?

If we were to assume a constant hazard rate for each component, then they would be estimated by dividing the total number of failure by the total test time.

$$\hat{\lambda} = \frac{n}{T} = \frac{n_1}{\left[\sum_{i=1}^{n_1} t_i\right] + T_0(n - n_1)}$$

where n_1 is the number of items that have failed (where the i^{th} failed item fails at time t_i), n is the total number of items test, and T_0 is the time at which all other items survived until.

For component 1:

$$\hat{\lambda}_1 = \frac{n_1}{\left[\sum_{i=1}^{n_1} t_i\right] + T_0(n - n_1)}$$

$$= \frac{6}{[26 + 314 + 858 + 899 + 905 + 906] + 1000(20 - 6)}$$

$$= \frac{6}{17\,968} = 0.000334\ failures\ per\ hour$$

For component 2:

$$\hat{\lambda}_2 = \frac{n_1}{\left[\sum_{i=1}^{n_1} t_i\right] + T_0(n - n_1)}$$

$$= \frac{10}{\begin{bmatrix} 250 + 308 + 411 + 495 + 501 \\ +620 + 635 + 725 + 781 + 900 \end{bmatrix} + 1000(20 - 10)}$$

$$= \frac{10}{15\,626} = 0.000640\ failures\ per\ hour$$

Therefore, the system failure rate assuming a constant hazard rate for each component is:

$$\hat{\lambda}_{system} = \hat{\lambda}_1 + \hat{\lambda}_2 = 0.000334 + 0.000640 = 0.000974\ failures\ per\ hour$$