MATH 61: INTRODUCTION TO DISCRETE STRUCTURES
MIDTERM #2

INSTRUCTOR: SPENCER UNGER

Name: Solutions

ID # ____________________

Section ________

Good Luck! Be sure to justify your answers!

No calculators, books or notes are allowed.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
Total	75	

Be careful, there are problems on both sides of the paper!

Date: May 16, 2016.

(1) (15 points total)

(a) (7 points) Give a combinatorial proof of Pascal's identity.

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

See class notes or the book

This problem is continued on the next page.

(b) (8 points) Let $S \subseteq \{n \in \mathbb{N} \mid 1 \leq n \leq 35\}$ with $|S| = 8$. Show that there are subsets X and Y of S so that the sum of the elements of X is equal to the sum of the elements of Y.

Define $f : P(S) \longrightarrow \mathbb{N}$

by $f(X) =$ Sum of the elements of X

The size of the domain is $2^8 = 256$

The largest possible value of f is $35+34+33+32+31+30+29+28$

$= 252$

So the size of the range is at most 252.

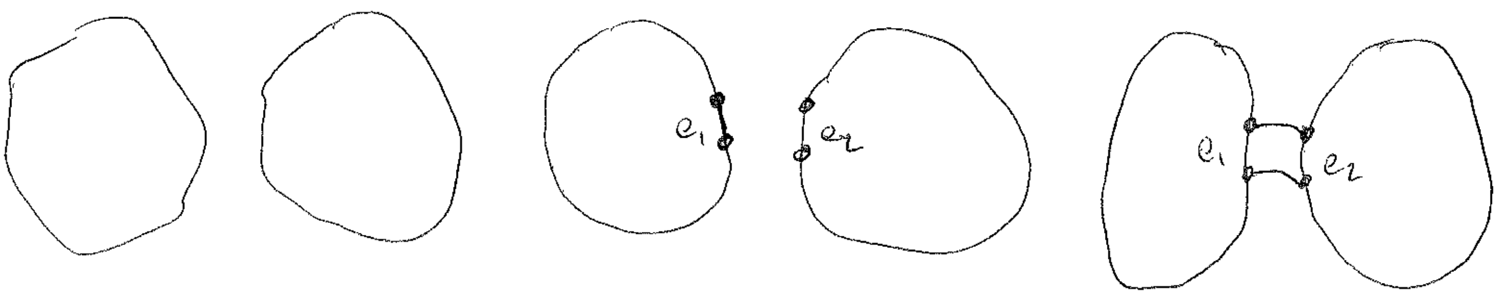
So by pigeonhole there are subsets $X, Y \subseteq S$

w.th $X \neq Y$ and $f(X) = f(Y)$

(2) (15 points) Answer the following questions.

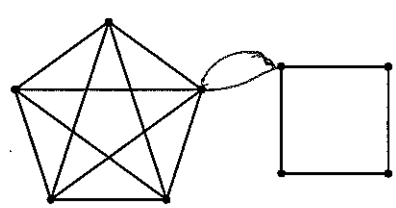
(a) Suppose G has exactly two connected components H_1 and H_2, and that each connected component has a Hamiltonian cycle. Call them p_1 and p_2. Now take $e_1 = \{u_1, v_1\}$ an edge appearing in p_1 and $e_2 = \{u_2, v_2\}$ an edge appearing in p_2. Show that if we add the edges $\{u_1, u_2\}$ and $\{v_1, v_2\}$ to G, then G has a Hamilton cycle.

Draw p_1 and p_2 Identify e_1 and e_2 Add edges



Removing e_1 and e_2 gives a Hamilton cycle.

(b) Below is a graph with two connected components. What is the smallest number of edges that you must add (adding parallel/multiple edges is allowed) to ensure that the graph has an Euler cycle? Exhibit this by adding edges to the graph and proving that the resulting graph has an Euler cycle.



All vertices have even degree so the graph has an Euler cycle by a theorem from class.

(3) (15 points total) Answer the following questions:

(a) (8 points) Let k be a positive odd number and m, n be positive numbers. How many paths of length k are there in $K_{m,n}$?

Let $k = 2l-1$. A path of length k is made by choosing $2l$ vertices in a row alternating sides of the graph.

Case 1 We start on the m side

$$m \cdot n \cdot \ldots \ldots m \cdot n = m^l n^l$$

Case 2 We start on the n side

$$n \cdot m \cdot \ldots \ldots n \cdot m = n^l m^l$$

Total is $2 m^l n^l$.

(b) (7 points) Give the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ subject to the constraints $x_1 \geq 9$ and $x_2, x_3, x_4, x_5 \geq 0$.

Replace x_1 with $x_1' = x_1 - 9$

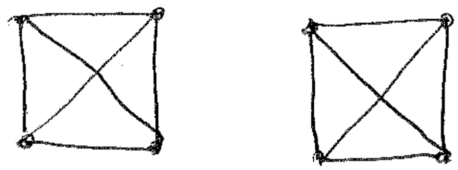
New equation is $x_1' + x_2 + x_3 + x_4 + x_5 = 21$ subject to $x_1', x_2, x_3, x_4, x_5 \geq 0$

So # of solutions is

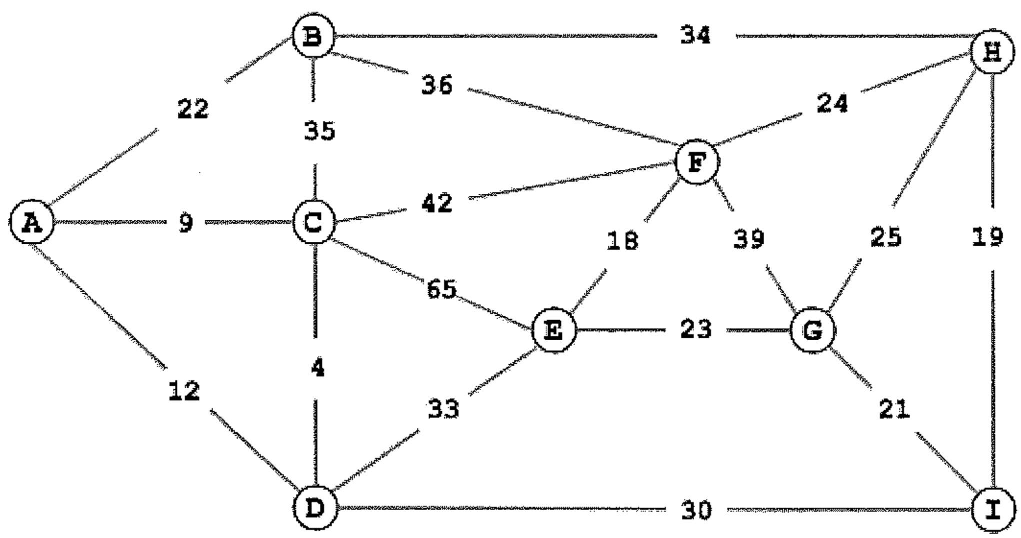
$$\binom{5+21-1}{5-1} = \binom{25}{4}.$$

(4) (15 points total)

(a) (7 points) Draw a graph which is not connected and every vertex has degree 3.



(b) (8 points) State the order that the shortest path algorithm visits the vertices while finding the length of the shortest path from A to F. **You do not need to show any work**



A, C, D, B, I, E, F

(b) Consider the following situation. An evil secretary is hosting a conference with n people each of which has a different name. The secretary makes name tags for each of the people attending, but refuses to give anyone their own name tag. Define S_n to the number of ways that the secretary can distribute name tags (one to each person) so that no one gets their own name tag. This question will have you compute a recurrence for S_n. **Your answers for parts (ii), (iii) and (iv) should involve S_k for $k < n$!**

(i) (2 points) What are S_1 and S_2?

$$S_1 = 0$$
$$S_2 = 1$$

(ii) (3 points) Consider the case where one of the people, Alice, has her name tag switched with someone else's. More formally, count the number of ways the secretary can distribute name tags to the n people where there is a person Z so that Z gets Alice's name tag and Alice gets Z's name tag.

$$(n-1) \cdot S_{n-2}$$

Choose who Z is

Distribute the rest of the name tags.

(iii) (3 points) Consider the case where the person who gets Alice's name tag does not have their name tag given to Alice. More formally, count the number of ways the secretary can distribute the name tags to the n people where there is a person Z so that Z gets Alice's name tag, but Alice does not get Z's name tag.

$$(n-1) \cdot S_{n-1}$$

Choose Z

Distribute the rest of the name tags. We have n-1 tags to distribute and each has 1 person it cannot go to.

(iv) (2 points) Using the previous two parts of the problem, complete a recurrence relation for S_n. If you don't know the answer to parts (ii) or (iii), then give your answer in terms of variables x and y where x is the answer for (ii) and y is the answer for (iii).

$$S_n = (n-1) S_{n-2} + (n-1) \cdot S_{n-1}$$

(5) (15 points total) Answer the following problems.

(a) (5 points) Solve the following recurrence relation:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with the initial conditions $a_0 = 1$ and $a_1 = 2$.

Considr $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

So $r = 3$

The general solution is

$a_n = x3^n + ny3^n$

$1 = a_0 = x + 0 \cdot y \longrightarrow x = 1$

$2 = a_1 = 3x + 3y \longrightarrow 2 = 3 + 3y$

So $y = -\frac{1}{3}$

$a_n = 3^n + -\frac{n}{3} \cdot 3^n$.

This problem is continued on the next page.