Math 61: Introduction to Discrete Structures
Midterm #2

Instructor: Spencer Unger

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Name: Solutions

ID # __________

Section __________

Good Luck! Be sure to justify your answers!

No calculators, books, or notes are allowed.

Problem	Points	Score
1	10	
2	20	
3	10	
4	10	
5	10	
6	20	
7	20	
Total	100	

1. (10 points) Solve the following recurrence relation:

The other version had different numbers.

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with the initial conditions $a_0 = 1$ and $a_1 = 2$.

Write $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$a_n = X 3^n + yn3^n$

$1 = a_0 = X 3^0 + y \cdot 0 \cdot 3^0 \longrightarrow X = 1$

$2 = a_1 = 3x + y \cdot 1 \cdot 3^1 \longrightarrow 2 = 3 + 3y$

$y = -\frac{1}{3}$

So $a_n = 3^n + \frac{-1}{3} \cdot n \cdot 3^n$

The other version had a 11 here.

2. (20 points) Let b_n be the number of binary strings of length n that *contain* 00. Derive a recurrence for b_n.

Case 1: String starts with 1

There are b_{n-1} ways to fill in the rest to ensure that there is 00 somewhere

Case 2: String starts with 01

There are b_{n-2} ways to fill in the rest to ensure that there is a 00.

Case 3: String starts with 00.

We can fill in any string of length $n-2$ since 00 is already there. 2^{n-2} ways to do this.

By the rule of sums

$$b_n = b_{n-1} + b_{n-2} + 2^{n-2}$$

Also, $b_0 = 0$, $b_1 = 0$

3. (10 points) Give an argument that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Consider $\{ X \mid X \subseteq \{1, 2, \cdots n+1\} \text{ and } |X| = k \}$

Count it two different ways.

First, $\binom{n+1}{k}$ this was why we defined the binomial coefficients.

Second, Case 1: $1 \in X$.

There are $\binom{n}{k-1}$ subsets of $\{1, 2, \cdots n+1\}$ size k with 1 as an element since we need to choose the remaining $k-1$ elements from an n element set.

Case 2: $1 \notin X$

There are $\binom{n}{k}$ such subsets since we need to choose k elements from an n element set $\{2, 3, \cdots n+1\}$.

So the total is $\binom{n}{k-1} + \binom{n}{k}$.

The other version featured a non-vegetarian option with different numbers.

4. (10 points) There are 47 people at a circular table. 24 people will be served tofu and 23 will be served tempeh.

   (a) Show that two people sitting next to each other will both get tofu.

Number positions around the table $0, 1, 2, \ldots 46$

Let $T_1 \ldots T_{24}$ be the positions of the people served tofu.

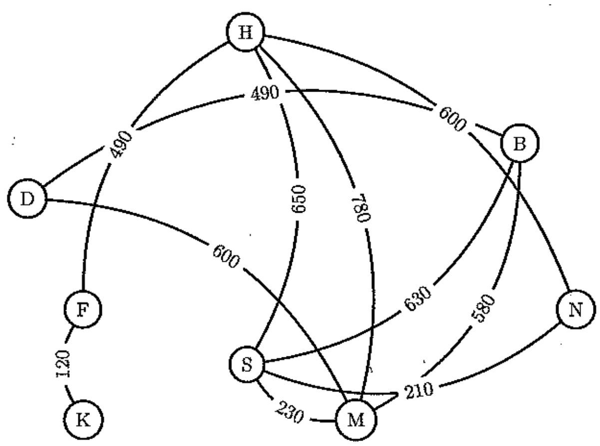
Let $$U_i = \begin{cases} U_i + 1 & \text{if } U_i < 46 \\ 0 & \text{if } U_i = 46 \end{cases}$$

We have 48 numbers $T_1 \ldots T_{24}, U_1 \ldots U_{24}$ between 0 and 46. So by Pigeonhole there are i, j with $T_i = U_j$.

   (b) Explain a configuration of the table where *no two* people sitting next to each other both get tempeh.

Alternate positions getting tempeh with positions getting tofu. The fact that there are more positions getting tofu means that no two tempeh eaters will sit next to each other.

5. (10 points) Write the order in which the shortest path algorithm visits the vertices of the graph when finding the shortest path from F to D.



F, K, H, N, S, M, B, D

6. (20 points) Suppose G is the graph with the following adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The other version had a different matrix that generated the same graph.

with the usual labelling of vertices from 1 to 8. Answer the following questions about G.

(a) How many paths of length 2 are there from vertex 1 to vertex 6?

We want the 1,6 entry of A^2. So

$$(0\,0\,0\,0\,0\,1\,1\,1) \cdot (1\,1\,1\,1\,1\,0\,0\,0) = 0$$

(b) We already have a name for the graph G. What is the name?

$K_{3,5}$

(c) Does G have an Euler cycle? Why or why not?

No. There are many vertices of odd degree.

So it cannot have one by our theorem from class.

(d) Does G have a Hamilton cycle? Why or why not?

No. A Hamilton cycle must alternate between sides of the bipartition. This means the sides must have the same size to have an H-cycle.

But they don't.

7. (20 points) Define a graph $G = (V, E)$ with $V = \mathbb{Z} \times \mathbb{Z}$ and $\{(w, x), (y, z)\} \in E$ exactly when $\max(|w - y|, |x - z|) = 5$.

(a) Just answer true or false for each:

i. $\{(3, 4), (4, 9)\} \in E$ T

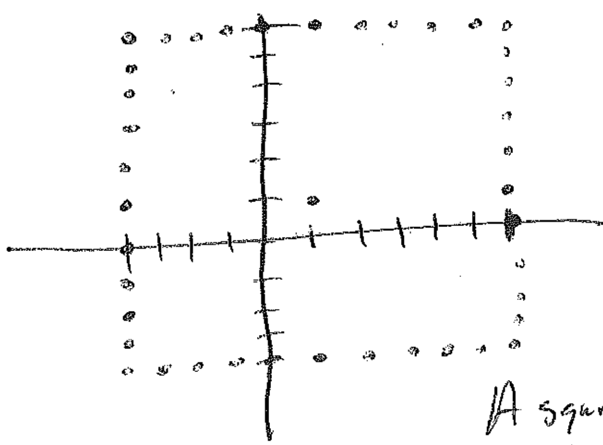
ii. $\{(8, 13), (4, 9)\} \in E$ F

iii. if $\{w, x), (y, z)\} \in E$, then the distance in the plane from (w, x) to (y, z) is less than 7. F

iv. There is a path of length 3 from $(0, 0)$ to $(1, 13)$. T

v. There are 4 vertices in V with all possible edges between them in E. T

(b) Determine the elements of the set $\{(w, x) \mid \{(1, 1), (w, x)\} \in E\}$. Drawing a picture is fine.



A square of side length 10 centered at $(1,1)$

(c) Prove or disprove G is connected. It is true.

Using the idea of part (b) repeatedly, a point p is connected to every point q on a square of side length $10 \cdot n$ centered at p.

So given points p and p' find k and l so that the square centered at p of side length $10 \cdot k$ overlaps the square centered at p' of side length $10 \cdot l$.

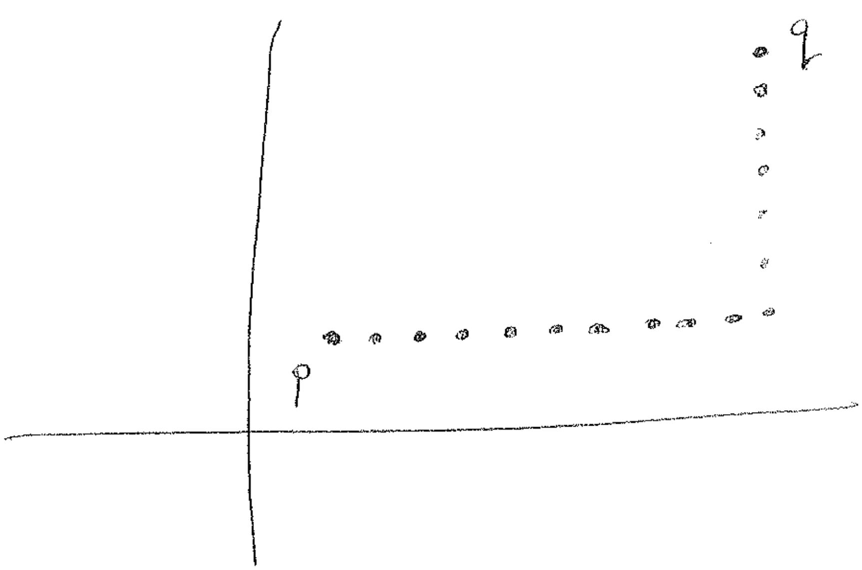
Then p and p' are connected by a path the goes through a vertex on the intersection of the two squares.

7 (c) argument 2

Given a point (x, y), we have the following edges

$\{(x,y), (x+1, y+5)\} \in E$ and $\{(x+1, y+5), (x+1, y)\} \in E$

So (x,y) and $(x+1,y)$ are connected by a path of Length 2. Similarly for (x,y) and $(x-1,y)$, $(x,y+1)$, $(x,y-1)$. So any two points are connected since any two adjacent vertices are connected by a path of length two in the following picture.



7 (c) Argument 3

Let (x,y) and (x',y') be vertices.

Set $x = 5n + k + x'$ where ~~$k \in [0,4]$~~ $0 \le k \le 4$

and $y = 5m + l + y'$ where $0 \le l \le 4$

It is easy to see that there is a path from

(x',y') to $(x'+5n, y')$ and from $(x'+5n, y')$ to

$(x'+5n, y'+5l)$.

Now $(x'+5n, y'+5l)$ and (x,y) ~~are in the~~

~~Same~~ have overlapping ~~...~~ squares using part (b), so

they are connected.